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Minimax-Based Selection, Adjustment and Disaggregation for Incompletely Specified Univariate Distributions

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(December 10, 1971)

This paper deals with making a nominal choice from a class of discrete univariate probability distributions about which one has "incomplete information" such as componentwise bounds, a componentwise ranking, or both. In some cases an initial distribution is provided, to be "adjusted" so as to be compatible with the incomplete information. The first part of the analysis systematizes and unifies the contents of four earlier papers treating such problems using a minimax-error or minimax-adjustment approach. The second part applies the same approach to situations in which an "aggregate" of the desired distribution is stipulated, either exactly, or approximately by componentwise bounds. All problems discussed in the paper can be formulated as linear programs, but relatively explicit solution methods are sought; unresolved difficulties arising in this attempt are identified.

Key words: Linear programs; mathematical models; minimax estimation; operations research; probability distribution.

1. Introduction

In the mathematical modeling efforts associated with an operations research study, it may be necessary to attribute numerical values to the probabilities of the various possible outcomes of some chance event, even though the information at hand is not sufficient to determine this probability distribution uniquely. Under these circumstances, it seems useful to have a systematic and reproducible method for selecting some one distribution from among those compatible with the available data, preferably for use as a "base-point" or "nominal case" for a subsequent sensitivity analysis.[1]

The paper which follows is the fifth in a series exploring one approach to this problem area. As before, attention is confined to discrete univariate distributions;[2] such a distribution will be represented by a real n-vector $\mathbf{x}$ which is a **probability vector,** i.e., its components x_i are nonnegative and sum to unity. The "incomplete information" about the "true" distribution is summarized as a constraint $\mathbf{x}\epsilon P$, where P is a closed bounded subset of n-space.

By the *minimax error selection problem* for P, we mean the problem of choosing $\mathbf{x}\epsilon P$ to minimize

$$F(\mathbf{x}) = \max \{\max_i | x_i - y_i | : \mathbf{y}\epsilon P\}. \tag{1.1}$$

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[1] NBS colleague R. C. Sansing, to whom (along with M. H. Pearl) I am indebted for a careful reading of the paper and for numerous helpful suggestions, points out that such a method might also prove useful to Bayesian data analysts in choosing a prior distribution consistent with the partial information at hand.

[2] A bivariate application will be given in [5].

The corresponding *weighted* problem for a vector $\mathbf{w}$ of positive "weights" $\{w_i\}_1^n$, is that of choosing $\mathbf{x}\epsilon P$ to minimize

$$F(\mathbf{x};\mathbf{w}) = \max\{\max_i w_i \mid x_i - y_i \mid : \mathbf{y}\epsilon P\}. \tag{1.2}$$

This corresponds to the situation in which errors in the attributed probabilities of some outcomes of the chance event are significantly more serious than errors in the attributed probabilities of other outcomes.

The analysis of such problems can be facilitated by noting that $F(\mathbf{x};\mathbf{w})$ admits the more explicit representation

$$F(\mathbf{x};\mathbf{w}) = \max_i \max\{w_i(M_i - x_i),\ w_i(x_i - m_i)\} \tag{1.3}$$

in terms of the quantities

$$M_i = \max\{y_i : \mathbf{y}\epsilon P\}, \tag{1.4}$$

$$m_i = \min\{y_i : \mathbf{y}\epsilon P\}. \tag{1.5}$$

This follows from the commutability of the three "maximum" operators in

$$F(\mathbf{x};\mathbf{w}) = \max_{\mathbf{y}\epsilon P} \max_i \max\{w_i(x_i - y_i), w_i(y_i - x_i)\}.$$

In view of (1.3), the weighted selection problem can be rephrased as that of selecting a number z and an n-vector $\mathbf{x}$ so as to minimize z, subject to constraints

$$z \geqslant w_i(M_i - x_i) \qquad (\text{all } i), \tag{1.6}$$

$$z \geqslant w_i(x_i - m_i) \qquad (\text{all } i), \tag{1.7}$$

$$\mathbf{x}\epsilon P. \tag{1.8}$$

If P is the solution-set of a finite family of linear equations (e.g., $\Sigma_i x_i = 1$) and/or inequalities (e.g., $x_i \geqslant 0$), then this problem is a *linear program* once the values of M_i and m_i are known; moreover M_i and m_i are themselves determinable as the extremal values of linear programs. All the particular cases described below can in fact be solved numerically by standard linear programming methods; however, we set a higher standard, expecting that the special form of these linear programs admits a solution algorithm which is more explicit (nearer to "closed-form"). The degree of explicitness demanded can best be understood by examining the previous papers [1-4],[3] or the material in section 2 of the present paper.

The information expressed by the condition $\mathbf{x}\epsilon P$ *may* be accompanied by information (e.g., data on some past analogous situation) suggesting an estimate of the desired distribution. We assume that this estimate, if present, is given as a probability n-vector $\mathbf{a}$, which is to be "adjusted as little as possible" to yield a distribution $\mathbf{x}\epsilon P$. This leads to the *minimax adjustment problem* for $(\mathbf{a}, P)$, of choosing $\mathbf{x}\epsilon P$ to minimize

$$G(\mathbf{x}) = \max_i \mid x_i - a_i \mid. \tag{1.9}$$

Again there is a more general *weighted* problem, that of choosing $\mathbf{x}\epsilon P$ to minimize

$$G(\mathbf{x};\mathbf{w}) = \max_i w_i \mid x_i - a_i \mid. \tag{1.9a}$$

Such a problem can in turn be rephrased as that of selecting a number z and an n-vector $\mathbf{x}$ so as to minimize z, subject to the constraints

$$z \geqslant w_i(x_i - a_i) \qquad (\text{all } i), \tag{1.10}$$

[3] Figures in brackets indicate the literature references at the end of this paper.

$$z \geq w_i(a_i - x_i) \qquad \text{(all } i\text{)}, \tag{1.11}$$

$$\mathbf{x} \epsilon P. \tag{1.12}$$

This too is a linear program when P is the solution-set of a finite family of linear equations and/or inequalities; once again the cases to be discussed are accessible numerically to linear programming techniques, but once again we seek more explicit solution algorithms.

The present paper has three distinguishable purposes. The *first* is to provide a more unified and systematic account of the material in [1-4] than was apparent when that material was first worked out. The *second* objective is to invite the attention of colleagues to some desired generalizations which as yet have proven recalcitrant in terms of the "more explicit solution algorithm" criterion mentioned above; the analysis of these generalizations will be carried here to the point where the mathematical difficulty becomes explicit, in the hope of stimulating others to resolve these difficulties.

The *third* goal is to apply this general methodology in a new area: that of disaggregating a given or "approximately given" probability distribution. The hypothesized situation here is that probabilities have been observed or estimated, for the outcomes of the chance event in question, according to some relatively *crude* classification of these outcomes; the problem at hand is to attribute numerical probabilities to the outcomes as specified in some *finer* classification. In other words, probabilities are given for "super-events" E_j each of which is a collection of mutually exclusive elementary events E_{ij}, and the aim is to specify the probabilities of the E_{ij}'s "optimally" in the sense of the minimax criterion used here. Readers familiar with "real-world" data-analysis and modeling problems will readily recognize how such problems can arise.

The following material is organized as follows: section 2 contains descriptions of, and status reports on, several subproblems to which our main problems can be reduced; section 3 both unifies and summarizes the cases satisfactorily treated in the four earlier papers, and also shows how the as-yet-unresolved generalizations of these cases reduce to certain of the subproblems presented in section 2.

In section 4, we consider the "minimax error selection" and "minimax adjustment" problems for the "disaggregation situation" described above, assuming that the distribution to be disaggregated is known *exactly*. The unweighted versions of these problems, and a restricted class of weighted versions, are shown to be solvable. Section 5 establishes the analogous results for the case in which the "crude" distribution is specified only in terms of upper and lower bounds on its components.

One further technical point should be noted. The problems described above will all be reduced to the form: choose number z and n-vector $\mathbf{x}$ to minimize z subject to certain constraints. In general, the extremal solution is not unique; along with $z_{\min}$ one has a convex polyhedron of optimizing $\mathbf{x}$'s rather than a single $\mathbf{x}$. A plausible "representative choice" from this polyhedron is the centroid of its vertices. The determination of this centroid is treated in some detail in section 5 of [1] and section 3 of [2], for the particular cases studied in those papers. These detailed treatments are regarded as adequate indications of how the analysis might go in other cases; thus the "centroid" approach to resolution of nonuniqueness is not taken up explicitly in [3] or [4] or the present paper.

2. Subproblems

In this section we describe several subproblems which will arise later, and report on their status as regards effective solution-methods. The reader may prefer to skip this section at first, coming back to it as the individual subproblems arise.

Problem I: Given m-vectors $\mathbf{A}$ and $\mathbf{B}$, and number S, find an m-vector $\mathbf{y}$ such that[4]

$$\mathbf{A} \leq \mathbf{y} \leq B, \tag{2.1}$$

$$\Sigma_k y_k = S. \tag{2.2}$$

[4] Inequalities between vectors are to be interpreted componentwise.

Obvious necessary conditions for a solution to exist are

$$\mathbf{A} \leq \mathbf{B}, \tag{2.3}$$

$$\Sigma_k A_k \leq S \leq \Sigma_k B_k. \tag{2.4}$$

These conditions are also sufficient. For if they hold, then when $\mathbf{A} = \mathbf{B}$ we can set $\mathbf{y} = \mathbf{A}$ and satisfy (2.1) and (2.2). When $\mathbf{A} \neq \mathbf{B}$ we can define

$$\Theta = [S - \Sigma_k A_k]/[\Sigma_k B_k - \Sigma_k A_k]$$

so that $0 \leq \Theta \leq 1$ by virtue of (2.4), and then satisfy (2.1)-(2.2) by putting

$$\mathbf{y} = (1 - \Theta)\mathbf{A} + \Theta\mathbf{B}. \tag{2.5}$$

Problem II. Given m-vectors $\mathbf{A}$ and $\mathbf{B}$, and number S, find an m-vector $\mathbf{y}$ satisfying not only (2.1)-(2.2), but also

$$y_1 \leq y_2 \leq \ldots \leq y_m. \tag{2.6}$$

Here it is convenient to define nondecreasing sequences $\{A_k'\}_1^m$ and $\{B_k'\}_1^m$, forming the components of respective vectors $\mathbf{A}'$ and $\mathbf{B}'$, by

$$A_k' = \max_{j \leq k} A_j, \qquad B_k' = \min_{j \geq k} B_j.$$

Obvious necessary conditions for a solution to exist are

$$\mathbf{A}' \leq \mathbf{B}' \qquad (\text{i.e., } A_i \leq B_j \text{ for } i \leq j), \tag{2.7}$$

$$\Sigma_k A_k' \leq S \leq \Sigma_k B_k' \tag{2.8}$$

That these conditions are also sufficient follows from the same construction as above, applied to $\mathbf{A}'$ and $\mathbf{B}'$.

Problem III: Given m-vector $\mathbf{Z}$, positive m-vector $\mathbf{v}$, and number S, find the minimum value z^* of z such that $z \geq 0$ and

$$\Sigma_k \max\{0, Z_k - z/v_k\} \leq S. \tag{2.9}$$

An obvious necessary condition for the existence of a solution is

$$S \geq 0. \tag{2.10}$$

That this condition is also sufficient is shown in section 2 of [4], where the following solution algorithm is justified subject to (2.10) as hypothesis.

First delete all $Z_k \leq 0$ from the problem; if no Z_k are left then $z^* = 0$. Suppose some *are* left; choose $Z_{m+1} = 0$ and any $v_{m+1} > 0$, and renumber so that

$$v_1 Z_1 \geq v_2 Z_2 \geq \ldots \geq v_m Z_m > v_{m+1} Z_{m+1} = 0. \tag{2.11}$$

The sequence $\{Z_k^*\}_1^{m+1}$ defined by

$$Z_k^* = \Sigma_1^k Z_j - v_k Z_k \Sigma_1^k (1/v_j) \tag{2.12}$$

is calculable by the recursion

$$Z_{k+1}^* = Z_k^* + (v_k Z_k - v_{k+1} Z_{k+1}) \Sigma_1^k (1/v_j)$$

and initial condition $Z_1^* = 0$. Unless $\Sigma_k Z_k < S$ (in which case $z^* = 0$), there is a unique $J \in \{1, 2, \ldots, m\}$ such that

$$0 = Z_1^* \leq Z_2^* \leq \ldots \leq Z_J^* < S \leq Z_{J+1}^* \leq \ldots \leq Z_{m+1}^*. \tag{2.13}$$

In terms of this J,

$$z^* = [\Sigma_1^J Z_k - S]/\Sigma_1^J (1/v_k). \tag{2.14}$$

Problem IV: Given m-vector $\mathbf{Z}$, positive m-vector $\mathbf{v}$ and number S, find the minimum value z^* of z such that $z \geq 0$ and

$$z - \Sigma_k \max \{0, Z_k - z/v_k\} \geq S. \tag{2.15}$$

This problem, not taken up explicitly in the earlier papers, will now be solved. The left-hand side of (2.15) is a continuous increasing function of z, bounded neither above nor below. The problem therefore always has a solution, and that solution is $z^* = 0$ if and only if the value of this function for $z = 0$ is $\geq S$. From now on we assume the opposite case,

$$S + \Sigma_k \max \{0, Z_k\} > 0; \tag{2.16}$$

in this case z^* is the unique value of z for which equality holds in (2.15).

Clearly all $Z_k \leq 0$ can be deleted from the problem. If none are left, then $z^* = S$; from now on we assume some left. Set $Z_{m+1} = 0$, choose any $v_{m+1} > 0$, and renumber so that

$$v_1 Z_1 \geq v_2 Z_2 \geq \ldots \geq v_m Z_m > v_{m+1} Z_{m+1} = 0 \tag{2.17}$$

Define a sequence $\{Z_k^*\}_1^{m+1}$ by

$$Z_k^* = v_k Z_k [1 + \Sigma_1^k (1/v_j)] - \Sigma_1^k Z_j; \tag{2.18}$$

it can be calculated from the initial condition $Z_1^* = v_1 Z_1$ and the recursion

$$Z_{k+1}^* = Z_k^* + (v_{k+1} Z_{k+1} - v_k Z_k) [1 + \Sigma_1^k (1/v_j)],$$

which shows the sequence to be nonincreasing. Since (2.16) with strict inequality (and all $Z_k > 0$) implies $S > Z_{m+1}^*$, there is a *least* $K \in \{1, 2, \ldots m+1\}$ for which $S > Z_K^*$.

If $z \leq v_K Z_K$, then

$$\begin{aligned} z - \Sigma_k \max \{0, Z_k - z/v_k\} &\leq z - \Sigma_1^K (Z_k - z/v_k) \\ &= z[1 + \Sigma_1^K (1/v_j)] - \Sigma_1^K Z_j \\ &\leq v_K Z_K [1 + \Sigma_1^K (1/v_j)] - \Sigma_1^K Z_j \\ &= Z_K^* < S, \end{aligned}$$

so that (2.15) is not satisfied. But if $K > 1$ and $v_K Z_K < z \leq v_{K-1} Z_{K-1}$, then (2.15) becomes

$$z - \Sigma_1^{K-1} (Z_k - z/v_k) \geq S,$$

which is equivalent to

$$z \geq z^* = [\Sigma_1^{K-1} Z_k + S]/[1 + \Sigma_1^{K-1} (1/v_k)]. \tag{2.19}$$

The value of z^* proposed in (2.19) is readily verified to satisfy $v_K Z_K \leq z^* \leq v_{K-1} Z_{K-1}$, and so is indeed the smallest $z \geq 0$ satisfying (2.15). Finally, if $K = 1$ so that $S > Z_1^* = v_1 Z_1$, then $v_1 Z_1 < z$ turns (2.15) into $z \geq S$, so that

$$z^* = S \qquad (\text{for } K = 1). \tag{2.20}$$

Problem V: Given m-vector $\mathbf{Z}$, positive m-vector $\mathbf{v}$ and number S, find the minimum value z^* of z such that $z \geq 0$ and

$$\Sigma_k \max \{0, \max_{j \leq k} (Z_j - z/v_j)\} \leq S. \tag{2.21}$$

We do not presently have a satisfactory solution method for this problem, not even under the additional assumption

$$0 < Z_1 \leq Z_2 \leq \ldots \leq Z_m \tag{2.22}$$

which will sometimes hold when the problem arises later. Note however that if **Z** and **v** are *similarly ordered*, in the sense that there exists a permutation π of $\{1,2,\ldots,m\}$ for which

$$Z_{\pi(1)} \leq Z_{\pi(2)} \leq \ldots \leq Z_{\pi(m)},$$

$$v_{\pi(1)} \leq v_{\pi(2)} \leq \ldots \leq v_{\pi(m)},$$

both hold, then the problem can be handled by previous methods. For in this case, for each $k \in \{1,2,\ldots,m\}$ there is a $p(k) \in \{1,2,\ldots,k\}$ such that

$$Z_{p(k)} = \max_{j \leq k} Z_j, \qquad v_{p(k)} = \max_{j \leq k} v_j,$$

so that for $z \geq 0$ (2.21) is equivalent to

$$\Sigma_k \max \{0, Z_{p(k)} - z/v_{p(k)}\} \leq S$$

and we are back to an instance of Problem III. This comment will, of course, in particular apply whenever either all Z_k are equal or all v_k are equal.

Problem VI: Given m-vector **Z**, positive m-vector **v** and number S, find the minimum value z^{**} of z such that $z \geq 0$ and

$$\Sigma_k \min_{j \geq k} (Z_j + z/v_j) \geq S. \tag{2.23}$$

Here too, a solution method is presently lacking in the general case. Suppose, however, that **Z** and **v** are *oppositely ordered*, in the sense that there exists a permutation π of $\{1,2,\ldots,m\}$ for which

$$Z_{\pi(1)} \leq Z_{\pi(2)} \leq \ldots \leq Z_{\pi(m)},$$

$$v_{\pi(1)} \geq v_{\pi(2)} \geq \ldots \geq v_{\pi(m)}$$

both hold. Then for each $k \in \{1,2,\ldots,m\}$, there is a $p(k) \in \{k, k+1, \ldots, m\}$ such that

$$Z_{p(k)} = \min_{j \geq k} Z_j, \qquad v_{p(k)} = \max_{j \geq k} v_j.$$

Then for $z \geq 0$, (2.23) is equivalent to

$$\Sigma_k (Z_{p(k)} + z/v_{p(k)}) \geq S,$$

so that in this case z^{**} is given by

$$z^{**} = \max \{0, [S - \Sigma_k Z_{p(k)}]/\Sigma_k (1/v_{p(k)})\}. \tag{2.24}$$

This of course applies, in particular, if all Z_k are equal or if all v_k are equal.

Problem VII: Given m-vectors **Y** and **Z**, positive m-vector **v** and number S, find the maximum value z^* of z such that $z \geq 0$ and

$$\Sigma_k \max \{Y_k, \max_{j \leq k} (Z_j - z/v_j)\} \leq S. \tag{2.25}$$

This has Problem V as a special case ($\mathbf{Y} = \mathbf{O}$), and so *also* at present lacks a satisfactory solution method except when **Z** and **v** are similarly ordered.

Resolution of the three Problems (V, VI, VII), presented so far as in the "unsolved" category,

would (as demonstrated later) yield solution methods for the "weighted" versions of the problems whose unweighted cases were solved in [2-4].

We turn now to problems associated with the new "disaggregation" considerations taken up in sections 4 and 5. These involve a partition of $\{1,2,\ldots,n\}$ into nonempty sets $\{S_j\}_1^m$. For any n-vector $\mathbf{x}$, and any subset T of $\{1,2,\ldots,n\}$, we set

$$\mathbf{x}(T)=\Sigma\{x_i: i\epsilon T\}=\Sigma_{i\epsilon T}x_i. \tag{2.26}$$

Problem VIII: Given n-vectors $\mathbf{A}$ and $\mathbf{B}$, m-vectors $\mathbf{s}^-$, and $\mathbf{s}^+$, and number S, find an n-vector $\mathbf{x}$ such that

$$\mathbf{A}\leq\mathbf{x}\leq\mathbf{B}, \tag{2.27}$$

$$\Sigma_i x_i=S, \tag{2.28}$$

$$s_j^-\leq\mathbf{x}(S_j)\leq s_j^+ \qquad \text{for } 1\leq j\leq m. \tag{2.29}$$

An obvious necessary condition for a solution, in view of (2.27), is

$$\mathbf{A}\leq\mathbf{B}. \tag{2.30}$$

Another is

$$\max\{s_j^-,\ \mathbf{A}(S_j)\}\leq\min\{s_j^+,\ \mathbf{B}(S_j)\} \qquad (1\leq j\leq m), \tag{2.31}$$

while a third is

$$\Sigma_j\ \max\{s_j^-,\ \mathbf{A}(S_j)\}\leq S\leq\Sigma_j\min\{s_j^+,\ \mathbf{B}(S_j)\}. \tag{2.32}$$

We now give a constructive proof that this trio of necessary conditions is also *sufficient*. Suppose the conditions hold. By (2.31) and (2.32), plus our analysis of Problem I, there is an m-vector $\mathbf{y}$ such that

$$\max\{s_j^-,\ \mathbf{A}(S_j)\}\leq y_j\leq\min\{s_j^+,\ \mathbf{B}(S_j)\} \qquad (1\leq j\leq m), \tag{2.33}$$

$$\Sigma y_j=S. \tag{2.34}$$

Since $\mathbf{A}(S_j)\leq y_j\leq\mathbf{B}(S_j)$, we can write

$$y_j=(1-\Theta_j)\mathbf{A}(S_j)+\Theta_j\mathbf{B}(S_j) \tag{2.35}$$

with $\Theta_j\epsilon[0,1]$. (If $\mathbf{A}(S_j)=\mathbf{B}(S_j)$, then Θ_j is arbitrary.)

Now define n-vector $\mathbf{x}$ by

$$x_i=(1-\Theta_j)A_i+\Theta_jB_i \qquad (i\epsilon S_j);$$

then (2.27) is satisfied since $\mathbf{A}\leq\mathbf{B}$ and $\Theta_j\epsilon[0,1]$. Since $\mathbf{x}(S_j)=y_j$, (2.28) and (2.29) follow from (2.34) and (2.33), respectively.

Problem IX: Given m-vector $\mathbf{Y}$, positive n-vector $\mathbf{w}$, n-vector $\mathbf{Z}$ and number S, find the minimum value z^* of z such that $z\geq 0$ and

$$\Sigma_j\max\{Y_j,\ \Sigma_{i\epsilon S_j}\max\{0,\ Z_i-z/w_i\}\}\leq S. \tag{2.36}$$

Note first that if $Z_i\leq 0$, then the associated summand in (2.36) vanishes. Thus we may assume all $Z_i>0$. Next, an obvious necessary condition for a solution to exist is

$$\Sigma_j\max\{Y_j,0\}\leq S; \tag{2.37}$$

this will be assumed in what follows. Finally, it will be assumed that

$$\Sigma_j\max\{Y_j,\ \Sigma_{i\epsilon S_j}Z_i\}>S, \tag{2.38}$$

since in the contrary case we would have $z^* = 0$.

Choose $Z_o = 0$ and arbitrary $w_o > 0$; renumber if necessary so that

$$0 = w_o Z_o < w_1 Z_1 \leq w_2 Z_2 \leq \ldots \leq w_n Z_n. \tag{2.39}$$

By (2.37) and (2.38), we have $0 < z^* < w_n Z_n$.

For $k \in \{1,2,\ldots,n\}$, set

$$S_{jk} = \{i : i \in S_j,\ i \geq k\} = S_j \cap \{k, k+1, \ldots, n\}, \tag{2.40a}$$

$$Z_{jk} = \Sigma\,\{Z_i : i \in S_{jk}\} = \mathbf{Z}(S_{jk}), \tag{2.40b}$$

$$1/v_{jk} = \Sigma\,\{1/w_i : i \in S_{jk}\}. \tag{2.40c}$$

Then for $z \in (w_{k-1}Z_{k-1},\ w_k Z_k]$, we have

$$[i \in S_j \text{ and } z \leq w_i Z_i] \quad \text{iff} \quad i \in S_{jk},$$

and thus

$$\Sigma_{i \in S_j} \max\,\{0,\ Z_i - z/w_i\} = Z_{jk} - z/v_{jk}.$$

Thus, $z \in (w_{k-1}Z_{k-1},\ w_k Z_k]$ satisfies (2.36) if and only if

$$\Sigma_j \max\,\{Y_j,\ Z_{jk} - z/v_{jk}\} \leq S,$$

which can be rewritten

$$\Sigma_j \max\,\{0,\ (Z_{jk} - Y_j) - z/v_{jk}\} \leq S - \Sigma_j Y_j. \tag{2.41}$$

This in turn can be true for some $z \in (w_{k-1}Z_{k-1},\ w_k Z_k]$, if and only if it is true for $z = w_k Z_k$, i.e.

$$\Sigma_j \max\,\{0,\ (Z_{jk} - Y_j) - w_k Z_k / v_{jk}\} \leq S - \Sigma_j Y_j. \tag{2.42}$$

These observations yield the following procedure for finding z^*. First, try $k = 1,2,\ldots,$ until reaching a first value for which (2.42) is satisfied. Second, with this value of k solve the instance (2.41) of Problem III. (We have not considered how the information $z^* \in (w_{k-1}Z_{k-1},\ w_k Z_k]$ might be used to streamline the second step.)

Problem X: Given m-vector $\mathbf{Y}$, n-vector $\mathbf{Z}$, positive n-vector $\mathbf{w}$, number S, and for each $j \in \{1,2,\ldots,m\}$ an enumeration of the $\sigma(j)$ members of S_j; write $w_i = w_r^j$ and $Z_i = Z_r^j$ if i is the rth member of S_j; find the least value z^* of z such that $z \geq 0$ and

$$\Sigma_j \max\,\{Y_j,\ \Sigma_{r=1}^{\sigma(j)} \max\,\{0,\ \max_{R \leq r}\,(Z_R^j - z/w_R^j)\}\} \leq S. \tag{2.43}$$

We presently lack a satisfactory solution method for this problem.

Problem XI: In the same setting as for Problem X, find the least value z^{**} of z such that $z \geq 0$ and

$$\Sigma_j \min\,\{Y_j,\ \min_{R \geq r}(Z_r^j + z/w_r^j)\} \geq S. \tag{2.44}$$

Here too, we still lack a satisfactory algorithm.

3. Previous Work and Unsolved Cases

3.1. Componentwise Bounds

A natural kind of "incomplete information," about a probability distribution arising in a practical context, consists of lower and upper bounds on the components of the probability n-vector

representing the distribution. If **L** and **U** denote the respective vectors of lower and upper bounds then the associated constraint set is the convex polyhedron

$$P = \{\mathbf{x}:\mathbf{x} \geqslant \mathbf{O},\ \Sigma_i x_i = 1,\ \mathbf{L} \leqslant \mathbf{x} \leqslant \mathbf{U}\}. \tag{3.1.1}$$

Replacing **L** by max $\{\mathbf{O},\mathbf{L}\}$ and **U** by min $\{\mathbf{U},\mathbf{1}\}$ if necessary, we can assume without loss of generality that $\mathbf{L} \geqslant 0$ and $\mathbf{U} \leqslant \mathbf{1}$. By the results for Problem I in section 2, the necessary and sufficient conditions for P to be nonempty are

$$\mathbf{L} \leqslant \mathbf{U}, \tag{3.1.2}$$

$$\Sigma_i L_i \leqslant 1 \leqslant \Sigma_i U_i; \tag{3.1.3}$$

these will now be assumed.

In [1], the weighted minimax error selection problem for this set P is solved. The following treatment, however, fits better with the present paper's framework. Observe that the problem can be transformed to that of choosing a number z and an n-vector **x** to minimize z subject to the constraints (1.6)-(1.8). In the present case, these constraints can be rewritten

$$\max\{L_i, M_i - z/w_i\} \leqslant x_i \leqslant \min\{U_i, m_i + z/w_i\} \quad \text{(all } i\text{)},$$

$$\Sigma_i x_i = 1.$$

For any fixed z, our results for Problem I show (constructively) that an associated **x** exists if and only if

$$\max\{L_i, M_i - z/w_i\} \leqslant \min\{U_i, m_i + z/w_i\} \quad \text{(all } i\text{)}, \tag{3.1.4}$$

$$\Sigma_i \max\{L_i, M_i - z/w_i\} \leqslant 1, \tag{3.1.5}$$

$$\Sigma_i \min\{U_i, m_i + z/w_i\} \geqslant 1. \tag{3.1.6}$$

The redundant constraint $z \geqslant 0$ can be imposed if desired.

Now the left-hand sides of (3.1.4) and (3.1.5) are nonincreasing in z, while the right-hand side of (3.1.4) and the left-hand side of (3.1.6) are nondecreasing. It follows that the desired minimum value of z can be written

$$z_{\min} = \max\{z^\circ, z^*, z^{**}\} \tag{3.1.7}$$

where we put

$$z^\circ = \text{minimum } z \text{ satisfying (3.1.4)}$$

$$z^* = \text{minimum nonnegative } z \text{ satisfying (3.1.5)},$$

$$z^{**} = \text{minimum nonnegative } z \text{ satisfying (3.1.6)}.$$

In view of (3.1.2), z° is well-defined, and is readily evaluated as

$$z^\circ = \max_i \max\{w_i(L_i - m_i), w_i(M_i - U_i), w_i(M_i - m_i)/2\}.$$

Since $L_i \leqslant m_i \leqslant M_i \leqslant U_i$, this reduces to

$$z^\circ = 1/2 \max_i\{w_i(M_i - m_i)\}. \tag{3.1.8}$$

By writing (3.1.5) as

$$\Sigma_i \max\{0, (M_i - L_i) - z/w_i\} \leqslant 1 - \Sigma_i L_i, \tag{3.1.9}$$

we see that the determination of z^* is an instance of Problem III in section 2, obeying the feasibility condition (2.10) by virtue of the left-hand half of (3.1.3).

Similarly, by rewriting (3.1.6) as

$$\Sigma_i \max \{0, (U_i - m_i) - z/w_i\} \leq \Sigma_i U_i - 1, \tag{3.1.10}$$

we see that the determination of z^{**} is also an instance of Problem III in section 2, obeying the feasibility condition (2.10) by virtue of the right-hand half of (3.1.3).

[Some calculation can be avoided on the basis of the following additional analysis. By (3.1.8), $-z^\circ/w_i \leq -(M_i - m_i)/2$, and so

$$\Sigma_i \max \{0, (M_i - L_i) - z^\circ/w_i\} \leq \Sigma_i \max \{0, (M_i + m_i)/2 - L_i\}$$
$$= \Sigma_i (M_i + m_i)/2 - \Sigma_i L_i,$$
$$\Sigma_i \max \{0, (U_i - m_i) - z^\circ/w_i\} \leq \Sigma_i \max \{0, U_i - (M_i + m_i)2\}$$
$$= \Sigma_i U_i - \Sigma_i (M_i + m_i)/2.$$

Comparing these relations with (3.1.9) and (3.1.10) respectively, we find that

$$z^\circ \geq z^* \quad \text{if } \Sigma_i (M_i + m_i) \leq 2, \tag{3.1.11a}$$

$$z^\circ \geq z^{**} \quad \text{if } \Sigma_i (M_i + m_i) \geq 2. \tag{3.1.11b}$$

Thus a preliminary evaluation of $\Sigma_i (M_i + m_i)$ will render unnecessary the determination of either z^* or z^{**}.]

Since determination of all three quantities compared in (3.1.7) can be regarded as "solved" problems, the same can be said for the desired minimization of z. There is only one gap, namely the explicit determination—for the set P specified in (3.1.1)—of the quantities M_i and m_i, defined by (1.4) and (1.5), which figure in determining z°, z^* and z^{**}.

For this purpose, note that for any i the condition $\mathbf{x} \epsilon P$ can be rewritten

$$L_i \leq x_i \leq U_i, \tag{3.1.12}$$

$$L_j \leq x_j \leq U_j \qquad (\text{all } j \neq i), \tag{3.1.13}$$

$$\Sigma_{j \neq i} x_j = 1 - x_i. \tag{3.1.14}$$

For any fixed x_i, the existence of an $(n-1)$-vector $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$ satisfying the last two conditions is shown by the results for Problem I (in sec. 2) to have the necessary and sufficient conditions $L_j \leq U_j$ for all $j \neq i$ (this is guaranteed by (3.1.2)) and

$$\Sigma_{j \neq i} L_j \leq 1 - x_i \leq \Sigma_{j \neq i} U_j. \tag{3.1.15}$$

Combining (3.1.12) and (3.1.15), we find that

$$M_i = \min \{U_i, 1 - \Sigma_{j \neq i} L_j\}, \tag{3.1.16}$$

$$m_i = \max \{L_i, 1 - \Sigma_{j \neq i} U_j\}. \tag{3.1.17}$$

This completes the analysis of the weighted minimax error *selection* problem, in the case where "incomplete information" is expressed as "componentwise bounds," so that the constraint set P is given by (3.1.1).

For this same case, we turn now to the minimax *adjustment* problem. The weighted version of this problem for $(\mathbf{a}, P)$—for *any* probability n-vector $\mathbf{a}$—is solved in section 3 of [4], using a treatment summarized below.

We know the problem can be transformed to that of choosing a number z and an n-vector $\mathbf{x}$ to minimize z subject to the constraints (1.10)-(1.12), which in the present case can be rewritten

$$\max \{L_i, a_i - z/w_i\} \leq x_i \leq \min \{U_i, a_i + z/w_i\} \qquad (\text{all } i),$$

$$\Sigma_i x_i = 1.$$

For any fixed z, our results for Problem I show (constructively) that an associated $\mathbf{x}$ exists if and only if

$$\max \{L_i, a_i - z/w_i\} \leq \min \{U_i, a_i + z/w_i\} \qquad \text{(all } i\text{)}, \tag{3.1.18}$$

$$\Sigma_i \max \{L_i, a_i - z/w_i\} \leq 1, \tag{3.1.19}$$

$$\Sigma_i \min \{U_i, a_i + z/w_i\} \geq 1. \tag{3.1.20}$$

The redundant constraint $z \geq 0$ can be imposed if desired.

As before, we can express the desired minimum value of z as

$$z_{\min} = \max \{z^\circ, z^*, z^{**}\},$$

where now

$$z^\circ = \text{minimum } z \text{ satisfying (3.1.18)},$$

$$z^* = \text{minimum nonnegative } z \text{ satisfying (3.1.19)},$$

$$z^{**} = \text{minimum nonnegative } z \text{ satisfying (3.1.20)}.$$

In view of (3.1.2), z° is well-defined, and is readily evaluated as

$$z^\circ = \max_i \max \{w_i(L_i - a_i), w_i(a_i - U_i)\}. \tag{3.1.21}$$

By writing (3.1.19) and (3.1.20) as

$$\Sigma_i \max \{0, (a_i - L_i) - z/w_i\} \leq 1 - \Sigma_i L_i,$$

$$\Sigma_i \max \{0, (U_i - a_i) - z/w_i\} \leq \Sigma_i U_i - 1,$$

we see that determining z^* and z^{**} are instances of Problem III in section 2, with satisfaction of the feasibility condition (2.10) ensured by (3.1.3). Thus the problem of minimizing z can be regarded as solved.

3.2. Componentwise Ranking

A *second* natural kind of "incomplete information," concerning a probability distribution arising in a practical context, would be a *ranking* of the probabilities of the various outcomes of the associated chance event. This translates into a componentwise ranking for the probability n-vector representing the distribution. Thus our constraint set here is given by

$$P = \{\mathbf{x}: 0 \leq x_1 \leq x_2 \leq \ldots \leq x_n, \Sigma x_i = 1\} \tag{3.2.1}$$

and is nonempty. As noted in section 2 of [2], it is easily verified that the quantities M_i and m_i of (1.4)-(1.5) are given in this case by

$$M_i = 1/(n+1-i) \qquad \text{(all } i\text{)}, \tag{3.2.2}$$

$$m_i = 0 \qquad (1 \leq i < n), \tag{3.2.3}$$

$$m_n = 1/n. \tag{3.2.4}$$

The weighted minimax error selection problem becomes that of choosing number z and n-vector $\mathbf{x}$ to minimize z subject to constraints (1.6)-(1.8), which here become[5]

$$\max \{0, M_1 - z/w_1\} \leq x_1 \leq z/w_1, \tag{3.2.5}$$

[5] We assume $n > 1$ throughout.

$$M_i - z/w_i \leq x_i \leq z/w_i \qquad (1 < i < n), \tag{3.2.6}$$

$$M_n - z/w_n \leq x_n \leq 1/n + z/w_n, \tag{3.2.7}$$

$$\Sigma_i x_i = 1, \tag{3.2.8}$$

$$x_1 \leq x_2 \leq \ldots \leq x_n. \tag{3.2.9}$$

Let us dispose first of the trivial case $n = 2$, when the constraints (3.2.5)-(3.2.9) become

$$\max\{0, \tfrac{1}{2} - z/w_1\} \leq x_1 \leq z/w_1, \tag{3.2.10}$$

$$1 - z/w_2 \leq x_2 \leq \tfrac{1}{2} + z/w_2, \tag{3.2.11}$$

$$x_1 + x_2 = 1, \tag{3.2.12}$$

$$x_1 \leq x_2. \tag{3.2.13}$$

These can be rewritten, in terms of x_1 alone, as

$$\max\{0, \tfrac{1}{2} - z/w_1, \tfrac{1}{2} - z/w_2\} \leq x_1 \leq \min\{\tfrac{1}{2}, z/w_1, z/w_2\},$$

which for $z \geq 0$ has a solution x_1 if and only if

$$\max\{\tfrac{1}{2} - z/w_1, \tfrac{1}{2} - z/w_2\} \leq \min\{z/w_1, z/w_2\}.$$

The smallest $z \geq 0$ satisfying this condition is

$$z_{\min} = \max\{w_1/4, w_2/4, [2(1/w_1 + 1/w_2)]^{-1}\},$$

which can be simplified to

$$z_{\min} = (1/4)\max\{w_1, w_2\}. \tag{3.2.14}$$

For $w_1 \geq w_2$, (3.2.14) and (3.2.10) yield $x_1 = 1/4$ when $z = z_{\min}$, while for $w_1 \leq w_2$, (3.2.14) and (3.2.11) yield $x_2 = 3/4$. Thus in either case the "optimal" $\mathbf{x}$ of the linear program is

$$\mathbf{x} = (1/4, 3/4) \qquad (n = 2). \tag{3.2.14a}$$

Now we assume $n > 2$. By the results for Problem II in section 2, for any fixed $z \geq 0$ the constraints (3.2.5)-(3.2.9) admit a solution n-vector $\mathbf{x}$ if and only if

$$1/(n+1-i) - z/w_i \leq z/w_j \qquad (1 \leq i \leq j < n), \tag{3.2.15}$$

$$1/(n+1-i) - z/w_i \leq 1/n + z/w_n \qquad (1 \leq i \leq n), \tag{3.2.16}$$

$$\Sigma_i \max\{0, \max_{j<i}[1/(n+1-j) - z/w_j]\} \leq 1, \tag{3.2.17}$$

$$(1/n + z/w_n) + \Sigma_{i<n} \min[1/n + z/w_n, \min_{i \leq j < n}(z/w_j)] \geq 1. \tag{3.2.18}$$

As before, the desired minimum value of z can be written

$$z_{\min} = \max\{z^\circ, z^*, z^{**}\}$$

where now

z° = minimum value of z satisfying (3.2.15)-(3.2.16),

z^* = minimum nonnegative value of z satisfying (3.2.17),

z^{**} = minimum nonnegative value of z satisfying (3.2.18).

It is easily verified that

$$z^\circ = \max\{\max_{i\leqslant j<n} 1/[(n+1-i)(1/w_i+1/w_j)],$$
$$\max_i (i-1)/[n(n+1-i)(1/w_i+1/w_n)]\}. \tag{3.2.19}$$

From this, we have

$$z^\circ/w_n \geqslant (n-1)/2n,\ z^\circ/w_j \geqslant 1/2(n+1-j) \qquad (j<n),$$

so that

$$(1/n+z^\circ/w_n)+\Sigma_{i<n}\min[1/n+z^\circ/w_n, \min_{i\leqslant j<n}(z^\circ/w_j)]$$
$$\geqslant (n+1)/2n+\Sigma_{i<n}\min[(n+1)/2n, 1/2\min_{i\leqslant j<n}(n+1-j)^{-1}]$$
$$=(n+1)/2n+\Sigma_{i<n}\min[(n+1)/2n, 1/2(n+1-i)]$$
$$=(n+1)/2n+\Sigma_{i<n}1/2(n+1-i)$$
$$=\tfrac{1}{2}[(1+1/n)+(1/2+1/3+\ldots+1/n)]>1;$$

comparison with (3.2.18) shows that $z^\circ \geqslant z^{**}$.

The determination of z^{**}, though not really necessary in view of the result in the previous paragraph, will be taken up next. Define n-vector $\mathbf{u}$ by

$$u_n = 1/w_n > 0,$$
$$u_{n-i} = 1/\max\{w_j : i \leqslant j < n\} \qquad (1 \leqslant i < n),$$

so that

$$u_1 \geqslant u_2 \geqslant \ldots \geqslant u_{n-1} > 0 \tag{3.2.20}$$

and the condition (3.2.18) which defines z^{**} (together with $z^{**} \geqslant 0$) is equivalent to

$$(1/n+zu_n)+\Sigma_{i<n}\min[1/n+zu_n, zu_i] \geqslant 1.$$

We rewrite this last condition as

$$z\Sigma_i u_1 - \Sigma_{i<n}\max\{z(u_i-u_n), 1/n\} \geqslant 0. \tag{3.2.21}$$

CASE 1. Suppose $u_n \geqslant u_1$, or equivalently, $w_n \leqslant \max\{w_i : i<n\}$. Then $u_n \geqslant u_i$ for $1 \leqslant i < n$. In view of the constraint $z \geqslant 0$ imposed on z^{**}, (3.2.21) becomes

$$z\Sigma_i u_i - (n-1)/n \geqslant 0,$$

leading to

$$z^{**} = (n-1)/n\Sigma_i u_i \qquad (\text{if } u_n \geqslant u_1). \tag{3.2.22}$$

CASE 2. If CASE 1 does *not* hold, there is a largest $J \in \{1,2,\ldots,n-1\}$ such that $u_J > u_n$. Then (3.2.21) becomes

$$z\Sigma_i u_i - (n-J-1)/n - \Sigma_1^J \max\{z(u_i-u_n), 1/n\} \geqslant 0,$$

which we rewrite as

$$z\{\Sigma_{J+1}^n u_i + Ju_n\} - \Sigma_1^J \max\{0, 1/n - z(u_i-u_n)\} \geqslant n-J-1)/n. \tag{3.2.23}$$

That we now have an instance of Problem IV solved in section 2, can be seen by setting

$$T=\Sigma_{J+1}{}^{n}u_i+Ju_n,$$

$$S=(n-J-1)/nT$$

$$Z_k=1/nT \qquad (1\leq k\leq J),$$

$$v_k=T/(u_k-u_n) \qquad (1\leq k\leq J).$$

Turning now to (3.2.17) we see that determining z^* is an instance of Problem V in section 2, for which a solution method is currently lacking. This then is the present *bottleneck* to solving the general weighted minimax error selection problem, for the case in which the "incomplete information" is represented by a componentwise ranking constraint on $\mathbf{x}$.

It *is*, however, possible to determine $z_{\min}$ under the special hypothesis

$$w_n=\max_i w_i. \tag{3.2.24}$$

In this case we have for $i\leq j<n$,

$$1/(n+1-i)\,(1/w_i+1/w_j)\leq w_n/2(n+1-i)\leq w_n/4\leq (n-1)w_n/2n,$$

and also

$$\max_i (i-1)/(n+1-i)\,(1/w_i+1/w_n)=(n-1)w_n/2,$$

so that (3.2.19) gives

$$z^\circ=(n-1)w_n/2n, \tag{3.2.25}$$

implying

$$\Sigma_i \max\{0,\ \max_{j\leq i}[1/(n+1-j)-z^\circ/w_j]\}$$

$$\leq \Sigma_i \max\{0, \max_{j\leq i}[1/(n+1-j)-(n-1)/2n]\}$$

$$=\Sigma_i \max\{0, 1/(n+1-i)-(n-1)/2n\}$$

$$=\Sigma_j \max\{0, 1/j-(n-1)/2n\}=\Sigma\{1/j-(n-1)/2n : 1\leq j\leq 2+2/(n-1)\}.$$

Considering separately the situations $n=3$ and $n\geq 4$, we find that the above expression equals $(1/2+1/n)\leq 1$, and comparison with (3.2.17) yields $z^\circ\geq z^*$. Since it has already been established that $z^\circ\geq z^{**}$, when (3.2.24) holds we have

$$z_{\min}=z^\circ=\tfrac{1}{2}(1-1/n)w_n. \tag{3.2.26}$$

With this value of z, (3.2.7) and (3.2.2) imply that each optimal $\mathbf{x}$ has

$$x_n=(n+1)/2n, \tag{3.2.26a}$$

"matching" (3.2.14a). These results agree with those obtained less systematically, for the unweighted case, in the earlier paper [2].

This completes the discussion of the "componentwise ranking" version of the mimimax error *selection* problem. We shift now to the same version of the minimax *adjustment* problem. The weighted case is that of choosing number z and n-vector $\mathbf{x}$ to minimize z subject to the constraints (1.10)-(1.12), which here become

$$a_i-z/w_i\leq x_i\leq a_i+z/w_i \qquad (\text{all } i), \tag{3.2.27}$$

$$\Sigma_i x_i=1, \tag{3.2.28}$$

$$0\leq x_1\leq x_2\leq \ldots\leq x_n. \tag{3.2.29}$$

For any fixed z, our results for Problem II in section 2 show (constructively) that an associated $\mathbf{x}$ exists if and only if

$$a_i - z/w_i \leq a_j + z/w_j \qquad \text{for } i \leq j, \tag{3.2.30}$$

$$\Sigma_i \max\{0,\ \max_{j<i}(a_j - z/w_j)\} \leq 1, \tag{3.2.31}$$

$$\Sigma_i \min_{j \geq i}(a_j + z/w_j) \geq 1. \tag{3.2.32}$$

As before, we can write

$$z_{\min} = \max\{z^\circ, z^*, z^{**}\},$$

where now

z° = smallest value of z satisfying (3.2.30),

z^* = least nonnegative z satisfying (3.2.31),

z^{**} = least nonnegative z satisfying (3.2.32).

By a simple evaluation,

$$z^\circ = \max_{i<j}\{(a_i - a_j)/[1/w_i + 1/w_j]\}. \tag{3.2.33}$$

The determination of z^* is an instance of Problem V in section 2, for which we *lack* a solution method (unless $\mathbf{a}$ and $\mathbf{w}$ are similarly ordered). The determination of z^{**}, in turn is an instance of section 2's Problem VI, for which a solution method is *also* lacking (unless $\mathbf{a}$ and $\mathbf{w}$ are oppositely ordered).

For the *unweighted* case (all $w_j = 1$), as noted in section 2, Problem V reduces to the solvable Problem III, so that z^* can be found; similarly finding z^{**} is an instance of the solvable "oppositely ordered" special case of Problem VI. Thus, as found in [4], the "componentwise ranking" version of the minimax adjustment problem is solvable in the unweighted case.

3.3 Both Componentwise Bounds and Ranking

In this subsection we consider the situation in which the "incomplete information" involves *both* componentwise bounds *and* a componentwise ranking on the probability n-vector in question. Thus our constraint set here is given by

$$P = \{\mathbf{x}: \mathbf{L} \leq \mathbf{x} \leq \mathbf{U}, x_1 \leq x_2 \leq \ldots \leq x_n, \Sigma_i x_i = 1\}.$$

It will be convenient to define n-vectors $\mathbf{L}'$ and $\mathbf{U}'$ by

$$L_i' = \max_{j<i} L_j,\ U_i' = \min_{j \geq i} U_j. \tag{3.3.1}$$

Then it is easy to see that

$$P = \{\mathbf{x}: \mathbf{L}' \leq \mathbf{x} \leq \mathbf{U}', x_1 \leq x_2 \leq \ldots \leq x_n, \Sigma_i x_i = 1\}. \tag{3.3.2}$$

Replacing L_i' by $\max\{0, L_i'\}$ and U_i' by $\min\{U_i', 1\}$ if necessary, we may assume that $\mathbf{0} \leq \mathbf{L}'$ and $\mathbf{U}' \leq \mathbf{1}$.

The analysis of Problem II in section 2 shows that P is nonempty if and only if

$$\mathbf{L}' \leq \mathbf{U}', \tag{3.3.3}$$

$$\Sigma_i L_i' \leq 1 \leq \Sigma_i U_i'. \tag{3.3.4}$$

These two conditions will be assumed to hold from now on.

For the above polyhedron P, and a fixed $i \in \{1,2,\ldots,n\}$, we proceed as in [3] to seek the quantities M_i, m_i of (1.4) and (1.5). Consider the conditions under which

$$\mathbf{x} = (y_1, \ldots, y_{i-1}, x_i, y_i, \ldots, y_{n-1}) \in P.$$

These conditions are

$$L_i' \leq x_i \leq U_i', \tag{3.3.5}$$

$$L_j' \leq y_j \leq \min\{U_j', x_i\} \qquad \text{for } j < i, \tag{3.3.6}$$

$$\max\{L_{j+1}', x_i\} \leq y_j \leq U_{j+1}' \qquad \text{for } i \leq j < n, \tag{3.3.7}$$

$$\Sigma_j y_j = 1 - x_i. \tag{3.3.8}$$

The discussion of Problem II in section 2 shows that an $(n-1)$-vector $\mathbf{y}$ satisfying (3.3.6) through (3.3.8) exists if and only if

$$L_j' \leq \min\{U_j', x_i\} \qquad \text{for } j < i, \tag{3.3.9}$$

$$\max\{L_{j+1}', x_i\} \leq U_{j+1}' \qquad \text{for } i \leq j < n, \tag{3.3.10}$$

$$\Sigma_{j<i} L_j' + \Sigma_{j=i}^{n-1} \max\{L_{j+1}', x_i\} \leq 1 - x_i, \tag{3.3.11}$$

$$\Sigma_{j<i} \min\{U_j', x_i\} + \Sigma_{j=i}^{n-1} U_{j+1}' \geq 1 - x_i. \tag{3.3.12}$$

Here (3.3.9) and (3.3.10) are automatically satisfied by virtue of (3.3.5). We can rewrite (3.3.11) and (3.3.12), respectively, as

$$x_i + \Sigma_{j=i}^{n-1} \max\{L_{j+1}', x_i\} \leq 1 - \Sigma_{j<i} L_j', \tag{3.3.13}$$

$$x_i + \Sigma_{j<i} \min\{U_j', x_i\} \geq 1 - \Sigma_{j=i}^{n-1} U_{j+1}'. \tag{3.3.14}$$

The left-hand sides of (3.3.13) and (3.3.14) are continuous increasing functions of x_i, bounded neither above nor below. They are therefore equal to their right-hand sides for unique respective values x_i^+ and x_i^-. It follows that (3.3.13) and (3.3.14) are equivalent to $x_i^- \leq x_i \leq x_i^+$. By combination with (3.3.5), this yields

$$M_i = \min\{U_i', x_i^+\}, \quad m_i = \max\{L_i', x_i^-\}. \tag{3.3.15}$$

To determine M_i and m_i, it therefore suffices [6] to determine x_i^+ and x_i^-. For this purpose, rewrite (3.3.14) as

$$x_i - \Sigma_{j<i} \max\{0, U_j' - x_i\} \geq 1 - \Sigma_j U_j';$$

thus finding x_i^- is an instance of Problem IV in section 2, and hence can be regarded as a solved problem.

Next, rewrite (3.3.13) as

$$x_i + \Sigma_{j \geq i} \max\{0, x_i - L_j'\} \leq 1 - \Sigma_j L_j' = S. \tag{3.3.16}$$

Clearly any solution x_i has $x_i \leq S$, so that $z = S - x_i \geq 0$; finding the maximum value x_i^+ of the solutions x_i of (3.3.16) is therefore equivalent to finding the minimum $z_i \geq 0$ such that

$$z_i - \Sigma_{j \geq i} \max\{0, (S - L_j') - z_i\} \geq 0, \tag{3.3.17}$$

another instance of Problem IV. Thus the determination of M_i and m_i can be regarded as a solved problem.

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[6] If the left-hand side of (3.3.13) does not exceed the right-hand side when $x_i = U_i'$, then $M_i = U_i'$ and x_i^+ need not be found; similarly for (3.3.14), L_i' and m_i.

We turn now to the (weighted) minimax error selection problem, which by (1.6)-(1.8) is that of choosing number z and n-vector $\mathbf{x}$ to minimize z subject to

$$\max \{L_i', M_i - z/w_i\} \leq x_i \leq \min \{U_i', m_i + z/w_i\}, \quad \text{(all } i\text{)}$$

$$\Sigma_i x_i = 1,$$

$$x_1 \leq x_2 \leq \ldots \leq x_n.$$

For any fixed z, it follows from the analysis of Problem II that an $\mathbf{x}$ satisfying these three conditions exists if and only if

$$\max_{j<i} \max \{L_j', M_j - z/w_j\} \leq \min_{j>i} \min \{U_j', m_j + z/w_j\} \quad \text{(all } i\text{)}, \tag{3.3.18}$$

$$\Sigma_i \max_{j<i} \max \{L_j', M_j - z/w_j\} \leq 1, \tag{3.3.19}$$

$$\Sigma_i \min_{j>i} \min \{U_j', m_j + z/w_j\} \geq 1. \tag{3.3.20}$$

The constraint $z \geq 0$ can be imposed if desired.

Following our customary pattern, set

$$z^\circ = \text{least } z \text{ satisfying (3.3.18)},$$

$$z^* = \text{least } z \geq 0 \text{ satisfying (3.3.19)},$$

$$z^{**} = \text{least } z \geq 0 \text{ satisfying (3.3.20)};$$

then we have for the desired minimum value of z,

$$z_{\min} = \max \{z^\circ, z^*, z^{**}\}.$$

Since (3.3.18) can be rewritten

$$\max \{L_i', \max_{j<i}(M_j - z/w_j)\} \leq \min \{U_i', \min_{j>i}(m_j + z/w_j)\},$$

it is readily found that

$$z^\circ = \max_{i \leq j} \{w_j(L_i' - m_j),\ w_i(M_i - U_j'),\ (M_i - m_j)/(1/w_i + 1/w_j)\}. \tag{3.3.21}$$

Since $L_i' \leq m_i$, while $i \leq j$ implies $L_i' \leq L_j' \leq m_j$, we have

$$(w_i + w_j)L_i' \leq w_i M_i + w_j m_j,$$

or equivalently

$$(w_i + w_j)(L_i' - m_j) \leq w_i(M_i - m_j),$$

implying

$$w_j(L_i' - m_j) \leq (M_i - m_j)/(1/w_i + 1/w_j).$$

Similarly, since $m_j \leq U_j'$ while $i \leq j$ implies $M_i \leq U_i' \leq U_j'$, we have

$$w_i(M_i - U_j') \leq 0 \leq w_j(U_j' - m_j),$$

or equivalently

$$(w_i + w_j)(M_i - U_j') \leq w_j(M_i - m_j),$$

implying

$$w_i(M_i - U_j') \leq (M_i - m_j)/(1/w_i + 1/w_j).$$

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Thus (3.3.21) becomes

$$z^\circ = \max_{i<j} \{(M_i - m_j)/(1/w_i + 1/w_j)\}. \tag{3.3.22}$$

Next, (3.3.19) and (3.3.20) can respectively be rewritten

$$\Sigma_i \max \{L_i', \max_{j<i}(M_j - z/w_j)\} \leq 1, \tag{3.3.23}$$

$$\Sigma_i \min \{U_i', \min_{j>i}(m_j + z/w_j)\} \geq 1. \tag{3.3.24}$$

From (3.3.23), the determination of z^* is clearly an instance of Problem VII in section 2. Rewriting (3.3.24) as

$$\Sigma_i \max \{-U_i', \max_{j>i}[(-m_j) - z/w_j]\} \leq -1 \tag{3.3.25}$$

shows that determining z^{**} is again an instance of Problem VII.

Thus here the *bottleneck* is the lack of a satisfactory solution method for Problem VII. The unweighted case (all $w_j = 1$) is solvable, as in [3], since then Problem VII reduces to the solvable Problem III.

We turn now to the minimax adjustment problem, which here involves selecting a number z and an n-vector $\mathbf{x}$ to minimize z, subject to

$$\max \{L_i', a_i - z/w_i\} \leq x_i \leq \min \{U_i', a_i + z/w_i\} \quad \text{(all } i\text{)},$$

$$\Sigma_i x_i = 1,$$

$$x_1 \leq x_2 \leq \ldots \leq x_n.$$

For any given z, an $\mathbf{x}$ obeying these three conditions exists if and only if

$$\max_{j<i} \max \{L_j', a_j - z/w_j\} \leq \min_{j>i} \min \{U_j', a_j + z/w_j\} \quad \text{(all } i\text{)},$$

$$\Sigma_i \max_{j<i} \max \{L_j', a_j - z/w_j\} \leq 1,$$

$$\Sigma_i \min_{j>i} \min \{U_j', a_j + z/w_j\} \geq 1,$$

requirements which are respectively equivalent to

$$\max \{L_i', \max_{j<i}(a_j - z/w_j)\} \leq \min \{U_i', \min_{j>i}(a_j + z/w_j)\} \text{ (all } i\text{)}, \tag{3.3.26}$$

$$\Sigma_i \max \{L_i', \max_{j<i}(a_j - z/w_j)\} \leq 1, \tag{3.3.27}$$

$$\Sigma_i \min \{U_i', \min_{j>i}(a_j + z/w_j)\} \geq 1. \tag{3.3.28}$$

Defining z°, z^* and z^{**} in analogy with the cases treated previously, we will again have

$$z_{\min} = \max \{z^\circ, z^*, z^{**}\}.$$

It is readily verified that

$$z^\circ = \max_{i<j} \{w_j(L_i' - a_j), w_i(a_i - U_j'), (a_i - a_j)/(1/w_i + 1/w_j)\}. \tag{3.3.29}$$

If $\mathbf{a}$ satisfies the modified componentwise bounds, i.e. $\mathbf{L}' \leq \mathbf{a} \leq \mathbf{U}'$, then it follows that for $i \leq j$,

$$L_i' \leq a_i \text{ and } L_i' \leq L_j' \leq a_j,$$

implying

$$w_j(L_i' - a_j) \leq 0 \leq w_i(a_i - L_i')$$

and thus

$$w_j(L_i' - a_j) \leq (a_i - a_j)/(1/w_i + 1/w_j);$$

similarly

$$w_i(a_i - U_j') \leq w_j(U_j' - a_j),$$

leading to

$$w_i(a_i - U_j') \leq (a_i - a_j)/(1/w_i + 1/w_j).$$

Thus $\mathbf{L}' \leq \mathbf{a} \leq \mathbf{U}'$ implies that (3.3.29) simplifies to

$$z^\circ = \max_{i<j} \{(a_i - a_j)/(1/w_i + 1/w_j)\}. \tag{3.3.30}$$

The form of (3.3.27) shows that finding z^* is an instance of Problem VII, for which we lack a solution method unless $\mathbf{a}$ and $\mathbf{w}$ are similarly ordered. Rewriting (3.3.28) as

$$\Sigma_i \max \{ - U_i', \max_{j \geq i} [(- a_j) - z/w_j]\} \leq - 1 \tag{3.3.31}$$

shows that the determination of z^{**} is also an instance of Problem VII, for which we lack a solution method unless $\mathbf{a}$ and $\mathbf{w}$ are *oppositely* ordered. The unweighted case (all $w_j = 1$) is however exhibited as solvable, in accordance with [4].

4. Minimax Disaggregation of an Exact Distribution

4.1. Minimax-Error Selection

In this section and the next, we deal with a fixed partition $\{S_j\}_1^m$ of $\{1,2,\ldots,n\}$ into m sets S_j *with at least two members each* (i.e., $|S_j| \geq 2$). The notation

$$\mathbf{x}(T) = \Sigma \{x_i : i \epsilon T\}$$

will be employed; here $\mathbf{x}$ is an n-vector with components $\{x_i\}_1^n$, and T is a subset of $\{1,2,\ldots,n\}$.

The intended interpretation is that $\{1,2,\ldots,n\}$ indexes the outcomes of some chance event as classified in a relatively "fine" manner, whereas $\{1,2,\ldots,m\}$ indexes the outcomes as classified more grossly. More specifically, the "micro-outcomes" indexed by the members $i \epsilon S_j$ constitute in aggregate the jth "macro-outcome."

In the present section, we treat several versions of the following problem: Given an initial positive probability m-vector $\mathbf{s}$, with component s_j representing the probability of the jth macro-outcome, determine in a systematic way a probability n-vector $\mathbf{x}$, whose component x_i is to be regarded as the probability of the ith micro-outcome.

Suppose first that no further conditions are placed on $\mathbf{x}$. Then the appropriate constraint set for $\mathbf{x}$ is the polyhedron

$$P = \{\mathbf{x} : \mathbf{x} \geq 0, \mathbf{x}(S_j) = s_j \text{ for } 1 \leq j \leq m\}. \tag{4.1.1}$$

Our approach to finding a suitable $\mathbf{x}$ is to solve the weighted minimax selection problem for P, i.e., to determine $\mathbf{x} \epsilon P$ so as to minimize

$$F(\mathbf{x};\mathbf{w}) = \max_{\mathbf{y} \epsilon P} \max_i w_i |x_i - y_i|. \tag{4.1.2}$$

Let $\mathbf{x}_j$ and $\mathbf{y}_j$ denote variable $|S_j|$ vectors whose components are indexed by the members $i \epsilon S_j$, and in $\mathbf{x}_j$-space define the polyhedron

$$P_j = \{\mathbf{x}_j : \mathbf{x}_j \geq 0, \mathbf{x}_j(S_j) = s_j\}. \tag{4.1.3}$$

Then clearly P is the Cartesian product of the P_j's; if we put

$$F_j(\mathbf{x}_j;\mathbf{w}_j) = \max\{w_i|x_i - y_i| : i\epsilon S_j; \mathbf{y}_j\epsilon P_j\} \tag{4.1.4}$$

then it follows that

$$F(\mathbf{x};\mathbf{w}) = \max_j F_j(\mathbf{x}_j;\mathbf{w}_j). \tag{4.1.5}$$

The consequence is that the weighted minimax error selection problem for P *decomposes* into m independent subproblems, the jth of which is the weighted minimax error selection problem for P_j.

To solve the jth subproblem, it is convenient to observe that the $M_i = M_i(P)$ and $m_i = m_i(P)$ of (1.4) and (1.5) are given, for $i\epsilon S_j$, by

$$M_i = \max\{x_i : \mathbf{x}_j\epsilon P_j\}, \tag{4.1.6}$$

The analog of (1.3), namely

$$F_j(\mathbf{x}_j;\mathbf{w}_j) = \max_{i\epsilon S_j} \max\{w_i(M_i - x_i), w_i(x_i - m_i)\}, \tag{4.1.8}$$

is readily seen to hold. Thus the jth subproblem can be reduced to selecting a number z_j and an $|S_j|$-vector $\mathbf{x}_j$ to minimize z_j subject to

$$\max\{0, M_i - z_j/w_i\} \leq x_i \leq m_i + z_j/w_i \qquad (\text{all } i\epsilon S_j), \tag{4.1.9}$$

$$\mathbf{x}(S_j) = s_j, \tag{4.1.10}$$

$$\mathbf{x}_j\epsilon P_j. \tag{4.1.11}$$

For the particular P and P_j given by (4.1.1) and (4.1.3), the constraint (4.1.11) is redundant in the presence of (4.1.9) and (4.1.10). By the analysis of Problem I in section 2, for any given $z_j \geq 0$ an associated $\mathbf{x}_j$ exists if and only if

$$M_i - z_j/w_i \leq m_i + z_j/w_i, \tag{4.1.12}$$

$$\Sigma\{\max[0, M_i - z_j/w_i] : i\epsilon S_j\} \leq s_j, \tag{4.1.13}$$

$$\Sigma\{m_{ij} + z_j/w_i : i\epsilon S_j\} \geq s_j. \tag{4.1.14}$$

With the familiar formula

$$z_j^{\min} = \max\{z_j^\circ, z_j^*, z_j^{**}\},$$

we have

$$z_j^\circ = \tfrac{1}{2}\max\{w_i(M_i - m_i) : i\epsilon S_j\}. \tag{4.1.15}$$

The determination of z_j^* from (4.1.13) is an instance of (solvable) Problem III in section 2. And (4.1.14) yields directly

$$z_j^{**} = \max\{0, [s_j - \Sigma_{i\epsilon S_j} m_i]/\Sigma_{i\epsilon S_j}(1/w_i)\}. \tag{4.1.16}$$

Thus this problem can be regarded as solved, once the M_i and m_i are known. But for the present P_j, it is easy to check that

$$M_i = s_j \text{ and } m_i = 0 \qquad (i\epsilon S_j),$$

so that (4.1.15) yields

$$z^\circ = \max_j z_j^\circ = \tfrac{1}{2}\max_j\{s_j \max_{i\epsilon S_j}(w_i)\}, \tag{4.1.17}$$

while (4.1.16) yields

$$z^{**} = \max_j z_j^{**} = \max_j \{s_j / \Sigma_{i \epsilon S_j} (1/w_i)\}. \tag{4.1.18}$$

Using (4.1.8), we have

$$z_{\min} = \max \{z^\circ, z^{**}, \max_j z_j^*\} \tag{4.1.19}$$

with z° and z^{**} given by (4.1.17) and (4.1.18), and z_j^* the solution of Problem III applied to

$$\Sigma \{\max[0, s_j - z_j/w_i] : i \epsilon S_j\} \leq s_j. \tag{4.1.20}$$

It is not hard to show, however, that $z_j^{**} \leq z_j^\circ$, so that $z^{**} \leq z^\circ$ and (4.1.19) becomes

$$z_{\min} = \max \{z^\circ, \max_j z_j^*\} = \max_j \{z_j^\circ, z_j^*\}. \tag{4.1.21}$$

To consider the order relation between z_j° and z_j^*, note that $z_j^* \geq z_j^\circ$ holds if and only if the opposite of (4.1.20) is satisfied for $z_j = z_j^\circ$, i.e., if and only if

$$\Sigma \{\max [0, 1 - (\max_{k \epsilon S_j} w_k)/2w_i] : i \epsilon S_j\} \geq 1. \tag{4.1.22}$$

This will in particular be true, in the *semi-weighted case* when all $w_i (i \epsilon S_j)$ have the same value W_j, so that for each such j

$$z_j^{\min} = z_j^* = W_j s_j [1 - 1/|S_j|]. \tag{4.1.23}$$

Should this apply for all $j \epsilon \{1,2, \ldots, m\}$, we have

$$z_{\min} = \max_j W_j s_j [1 - 1/|S_j|], \tag{4.1.24}$$

which in particular holds for the unweighted version of the original problem.

We next treat the solvability of the weighted minimax selection problem under further restrictions on $\mathbf{x}$. To preserve the *decomposition* which guided the solution in the previous case, these constraints must maintain the independence of the m subproblems, i.e., must deal individually with the subvectors $\mathbf{x}_j$ of $\mathbf{x}$.

Consider *first* the case of componentwise bounds on $\mathbf{x}$, so that (4.1.1) is replaced by

$$P = \{\mathbf{x}: \mathbf{0} \leq \mathbf{L} \leq \mathbf{x} \leq \mathbf{U}, \mathbf{x}(S_j) = s_j \text{ for } 1 \leq j \leq m\}, \tag{4.1.25}$$

where $0 \leq \mathbf{L} \leq \mathbf{U}$ may be assumed at the outset. Let $\mathbf{L}_j$ and $\mathbf{U}_j$ be the subvectors of $\mathbf{L}$ and $\mathbf{U}$ associated with S_j. Then the jth subproblem has constraint set

$$P_j = \{\mathbf{x}_j : \mathbf{x}_j(S_j) = s_j, \mathbf{L}_j \leq \mathbf{x}_j \leq \mathbf{U}_j\}. \tag{4.1.26}$$

The weighted minimax selection problem for P_j can be solved as in subsection 3.1 of this paper; indeed, a rescaling by $1/s_j$ reduces it to the first problem treated in that subsection. The feasibility conditions, in addition to $\mathbf{L}_j \leq \mathbf{U}_j$, are given by

$$\mathbf{L}(S_j) = \Sigma \{L_i : i \epsilon S_j\} \leq s_j \leq \Sigma \{U_i : i \epsilon S_j\} = \mathbf{U}(S_j) \tag{4.1.27}$$

for the jth subproblem.

Second, a componentwise ranking might be imposed on each of the subvectors $\mathbf{x}_j$. The jth subproblem, after rescaling by $1/s_j$, is then of the type treated in subsection 3.2; in particular we have a solution method in the semiweighted case ($w_i = W_j$ for all $i \epsilon S_j$)—or more generally, if the ordering of the subvector $\mathbf{w}_j$ of $\mathbf{w}$ is consistent with the ranking imposed on $\mathbf{x}_j$.

Third, both componentwise bounds and a componentwise ranking might be imposed on each $\mathbf{x}_j$.

Rescaling then yields the type of problem taken up in subsection 3.3; again we have a solution method if $w_i = W_j$ for all $i \epsilon S_j$.

We can of course have "mixed" cases of the last three situations: some $\mathbf{x}_j$ may be subjected to componentwise bounds only, others to a ranking only, and still others to both. So long as the subproblems remain independent, one simply applies to each subproblem that part of the preceding discussion which is applicable.

4.2 Minimax Adjustment

Here the polyhedra P and P_j are as in the preceding subsection, but the function to be minimized is

$$G(\mathbf{x};\mathbf{w}) = \max_j G_j(\mathbf{x}_j;\ \mathbf{w}_j)\,, \tag{4.2.1}$$

where

$$G_j(\mathbf{x}_j;\mathbf{w}_j) = \max\ \{w_i\ |x_i - a_i| : i \epsilon S_j\} \tag{4.2.2}$$

and $\mathbf{a}$ is a given probability n-vector. Again we have m independent subproblems; the jth of them is to select number z_j and $|S_j|$-vector $\mathbf{x}_j$ so as to minimize z_j subject to

$$\max\ \{0,\ a_i - z_j/w_i\} \leqslant x_i \leqslant a_i + z_j/w_i \qquad (\text{all } i \epsilon S_j), \tag{4.2.3}$$

$$\mathbf{x}(S_j) = s_j, \tag{4.2.4}$$

$$\mathbf{x}_j \epsilon P_j. \tag{4.2.5}$$

The first case considered is that in which $\mathbf{x}$ is constrained only by the aggregation condition (4.2.4), so that (4.2.5) is redundant. For any $z_j \geqslant 0$, there exists an $\mathbf{x}_j$ satisfying (4.2.3) and (4.2.4) if and only if

$$a_i - z_j/w_i \leqslant a_i + z_j/w_i \qquad (\text{all } i \epsilon S_j), \tag{4.2.6}$$

$$\Sigma\ \{\max[0,\ a_i - z_j/w_i] : i \epsilon S_j\} \leqslant s_j, \tag{4.2.7}$$

$$\Sigma\ \{a_i + z_j/w_i : i \epsilon S_j\} \geqslant s_j. \tag{4.2.8}$$

Here (4.2.6) is automatically satisfied, so that with the familiar notation,

$$z_j^{\min} = \max\ \{z_j^*,\ z_j^{**}\}. \tag{4.2.9}$$

From (4.2.8) one obtains directly

$$z_j^{**} = \max\ \{0,\ s_j - \mathbf{a}(S_j)\}/\Sigma\{1/w_i : i \epsilon S_j\}. \tag{4.2.10}$$

The determination of z_j^*, as the minimum value of z_j satisfying both $z_j \geqslant 0$ and (4.2.7), is an instance of Problem III in section 2, and so can be regarded as solved. Note that if $s_j \geqslant \mathbf{a}(S_j)$, then (4.2.7) is satisfied by $z_j = 0$, so that (4.2.9) yields $z_j^{\min} = z_j^{**}$; on the other hand, if $s_j \leqslant \mathbf{a}(S_j)$, then $z_j^{\min} = z_j^*$.

The remaining cases are those in which $\mathbf{x}_j$ may be constrained by componentwise bounds, a componentwise ranking or both. Their treatment resembles that in the previous subsection, drawing on the analyses of minimax adjustment problems in subsections 3.1, 3.2 and 3.3. The case of componentwise bounds is solvable in general, but at present the other two cases admit satisfactory solution only when $w_i = W_j$ for all $i \epsilon S_j$.

5. Minimax Disaggregation of a "Fuzzy" Distribution

5.0 No Other Information

In section 4, it was assumed that a probability vector $\mathbf{s}$ describing the probabilities of the chance event's "macro-outcomes" is known *exactly*. This assumption played an essential technical

role in simplifying the analysis: the aggregation conditions $\mathbf{x}(S_j) = s_j$, together with the fact that $\Sigma_j s_j = 1$, implied that

$$\Sigma_i x_i = 1. \tag{5.0.1}$$

Thus it was not necessary to impose (5.0.1) as an explicit constraint; this condition, relating the m subvectors $\{\mathbf{x}_j\}_1^m$, would have ruined section 4's approach of dealing with m *independent* subproblems each involving a single $\mathbf{x}_j$.

For some situations, however, the assumption of an exactly known initial distribution $\mathbf{s}$ may well be unacceptable. In the present section, we replace the "exact" aggregation conditions $\mathbf{x}(S_j) = s_j$ by "approximate aggregation" conditions

$$s_j^- \leq \mathbf{x}(S_j) \leq s_j^+ \qquad (1 \leq j \leq m), \tag{5.0.2}$$

involving the components of given m-vectors $\mathbf{s}^-$ and $\mathbf{s}^+$ which (cf. Problem I in sec. 2) we may assume to satisfy

$$\mathbf{0} \leq \mathbf{s}^- \leq \mathbf{s}^+ \leq \mathbf{1}, \tag{5.0.3}$$
$$\Sigma_j s_j^- \leq 1 \leq \Sigma_j s_j^+.$$

This loosening means that the constraint (5.0.1) must be explicitly imposed; the m subproblems are no longer independent, but their interdependence is limited to the single condition (5.0.1).

The simplest weighted minimax-error selection problem of this type has

$$P = \{\mathbf{x}: \mathbf{x} \geq 0, \Sigma_i x_i = 1, \qquad s_j^- \leq \mathbf{x}(S_j) \leq s_j^+ \text{ for } 1 \leq j \leq m\}. \tag{5.0.4}$$

The first step in the analysis is to determine the M_i and m_i for this polyhedron P.

For a fixed $J \in \{1,2,\ldots,m\}$, a fixed $i \in S_J$, and a fixed $x_i \in [0,1]$, consider Problem VIII in section 2 with $S = 1$ and

$$A_i = B_i = x_i,$$
$$A_k = 0 \text{ and } B_k = 1 \text{ for } k \in \{1,2,\ldots,n\} - \{i\}.$$

Of the conditions for a solution to exist, (2.30) is satisfied automatically. Condition (2.31) is satisfied for $j \neq J$, while $j = J$ yields

$$\max\{s_J^-, x_i\} \leq \min\{s_J^+, x_i + |S_J| - 1\}. \tag{5.0.5}$$

Finally, (2.32) yields

$$\Sigma_{j \neq J} s_j^- + \max\{s_J^-, x_i\} \leq 1, \tag{5.0.6}$$

$$\Sigma_{j \neq J} s_j^+ + \min\{s_J^+, x_i + |S_J| - 1\} \geq 1. \tag{5.0.7}$$

M_i, as the largest value of x_i satisfying the last three conditions, is given by

$$M_i = M_J^* = \min\{s_J^+, 1 - \Sigma_{j \neq J} s_j^-\} \qquad (i \in S_J); \tag{5.0.8}$$

similarly we have

$$m_i = 0. \tag{5.0.9}$$

Next, (1.6)–(1.8) yield

$$\max\{0, M_i - z/w_i\} \leq x_i \leq z/w_i \qquad (\text{all } i),$$
$$\Sigma_i x_i = 1,$$
$$s_j^- \leq \mathbf{x}(S_j) \leq s_j^+ \qquad (\text{all } j).$$

By the analysis of Problem VIII in section 2, for any $z \geq 0$ an associated $\mathbf{x}$ exists if and only if

$$M_i - z/w_i \leq z/w_i \qquad \text{(all } i\text{)}, \tag{5.0.10}$$

$$\max \{s_j^-, \Sigma_{i \in S_j} \max \{0, M_i - z/w_i\}\} \leq \min \{s_j^+, z\Sigma_{i \in S_j}(1/w_i)\}, \tag{5.0.11}$$

$$\Sigma_j \max \{s_j^-, \Sigma_{i \in S_j} \max \{0, M_i - z/w_i\}\} \leq 1, \tag{5.0.12}$$

$$\Sigma_j \min \{s_j^+, z\Sigma_{i \in S_j}(1/w_i)\} \geq 1. \tag{5-0.13}$$

We wish to find the smallest $z \geq 0$ satisfying these four conditions.

The first condition, (5.0.10), yields

$$\begin{aligned} z \geq z^\circ &= \max_j z_j^\circ \\ &= \max_j [\tfrac{1}{2} M_j^* \max_{i \in S_j} w_i]. \end{aligned} \tag{5.0.14}$$

with M_j^* given by (5.0.8). The second condition, (5.0.11), is equivalent to the following trio of conditions. One is

$$z \geq z^{\circ\circ} = \max_j [s_j^- / \Sigma_{i \in S_j}(1/w_i)]. \tag{5.0.15}$$

A second is

$$\Sigma_{i \in S_j} \max \{0, M_j^* - z/w_i\} \leq s_j^+ \qquad \text{(all } j\text{)}. \tag{5.0.16}$$

If $z_j^{\circ\circ\circ}$ is the smallest $z \geq 0$ satisfying the jth of these requirements (5.0.16), then

$$z \geq z^{\circ\circ\circ} = \max_j z_j^{\circ\circ\circ}. \tag{5.0.17}$$

The determination of $z_j^{\circ\circ\circ}$ is a simple case ($\mathbf{Z}$ has equal components) of Problem III in section 2. The third "fragment" of (5.0.11) is

$$\Sigma_{i \in S_j} \max \{0, M_j^* - z/w_i\} \leq z\Sigma_{i \in S_j}(1/w_i) \qquad \text{(all } j\text{)}, \tag{5.0.18}$$

if $z_j^{\circ\circ\circ\circ}$ is the smallest $z \geq 0$ satisfying the jth of these, then

$$z \geq z^{\circ\circ\circ\circ} = \max_j z_j^{\circ\circ\circ\circ}.$$

The determination of $z_j^{\circ\circ\circ\circ}$ is an instance (again $\mathbf{Z}$ has equal components) of Problem IV in section 2.

The import of the last paragraph is that we have satisfactory solution methods for finding the least $z \geq 0$ satisfying (5.0.10) and (5.0.11). Before going on to (5.0.12) and (5.0.13), some additional comments can be made concerning this "solved" part of the problem. First, since

$$1/\Sigma_{i \in S_j}(1/w_i) \leq \tfrac{1}{2} \max_{i \in S_j} w_i,$$

and

$$s_j^- \leq s_j^+, \qquad s_j^- \leq 1 - \Sigma_{J \neq j} s_J^-,$$

it follows from (5.0.8), (5.0.14) and (5.0.15) that

$$z^{\circ\circ} \leq z^\circ. \tag{5.0.19}$$

Second, we will show that

$$z^{\circ\circ\circ\circ} \leq z^\circ, \tag{5.0.20}$$

by proving that

$$z_j^{\circ\circ\circ\circ} \leq z_j^\circ = \tfrac{1}{2} M_j^* \max_{i \in S_j} w_i.$$

Since $z_j^{\circ\circ\circ\circ}$ is defined in terms of (5.0.18), it suffices to show that (5.0.18) is satisfied for $z = z_j^{\circ}$, i.e., that

$$2z_j^{\circ}\Sigma_{i\epsilon S_j}(1/w_i) \geq \Sigma_{i\epsilon S_j} \max\{z_j^{\circ}/w_i, M_j^*\}.$$

This follows from a termwise comparison, the explicit expression for z_j° being used to show that $2z_j^{\circ}/w_i \geq M_j^*$.

We turn now to (5.0.12) and (5.0.13). Let z^{**} be the least $z \geq 0$ satisfying (5.0.13), which we rewrite as

$$\Sigma_j \max\{0, s_j^+ - z\Sigma_{i\epsilon S_j}(1/w_i)\} \leq \Sigma_j s_j^+ - 1. \tag{5.0.21}$$

Then the determination of z^{**} is an instance of Problem III in section 2, and so can be regarded as a solved problem.

Next, let z^* be the least $z \geq 0$ satisfying (5.0.12), so that the minimum value of z subject to (5.0.10)-(5.0.13) can be written

$$z_{\min} = \max\{z^{\circ}, z^{\circ\circ\circ}, z^{**}, z^*\}, \tag{5.0.22}$$

where $z^{\circ\circ}$ and $z^{\circ\circ\circ\circ}$ are ruled out of contention by (5.0.19) and (5.0.20), z° is given explicitly by (5.0.14), and the determination of $z^{\circ\circ\circ}$ and z^{**} are solved problems. The only remaining element in the solution of this weighted minimax-error selection problem is the determination of z^* from (5.0.12), which we rewrite as

$$\Sigma_j \max\{s_j^-, \Sigma_{i\epsilon S_j} \max\{0; M_j^* - z/w_i\}\} \leq 1 \tag{5.0.23}$$

with M_j^* given by (5.0.8).

This last element, however, is an instance of (solved) Problem IX in section 2, so that we have assembled a solution method for this weighted minimax selection problem.

A simpler solution method is at hand for what we have called the *semi-weighted case*, that in which

$$w_i = W_j \qquad \text{for all } i\epsilon S_j. \tag{5.0.24}$$

In that case (5.0.23) can be written as

$$\Sigma_j \max\{s_j^-, |S_j| \max\{0, M_j^* - z/W_j\}\} \leq 1,$$

which simplifies (since $s_j^- \geq 0$) to

$$\Sigma_j \max\{s_j^-, \ |S_j|(M_j^* - z/W_j)\} \leq 1$$

and thus to

$$\Sigma_j \max\{0, [|S_j|M_j^* - s_j^-] - z|S_j|/W_j\} \leq 1 - \Sigma_j s_j^-. \tag{5.0.25}$$

Hence determining z^* is an instance of Problem III in section 2. Moreover, (5.0.14) and (5.0.16) yield

$$z_j^{\circ} = \tfrac{1}{2}M_j^* W_j, \tag{5.0.26}$$

$$z_j^{\circ\circ\circ} = \max\{0, W_j[M_j^* - s_j^+/|S_j|]\}, \tag{5.0.27}$$

while z^{**} is the least $z \geq 0$ satisfying

$$\Sigma_j \max\{0, s_j^+ - z|S_j|/W_j\} \leq \Sigma_j s_j^+ - 1 \tag{5.0.28}$$

so that its determination is an instance of Problem III.

We turn now to the associated minimax adjustment problem. Relations (1.10)-(1.12) yield

$$\max\{0, a_i - z/w_i\} \leq x_i \leq a_i + z/w_i, \qquad \text{(all } i\text{)}$$

$$\Sigma_i x_i = 1,$$

$$s_j^- \leq \mathbf{x}(S_j) \leq s_j^+ \qquad \text{(all } j\text{)}.$$

This instance of Problem VIII has a solution if and only if

$$\max\{0, a_i - z/w_i\} \leq a_i + z/w_i \qquad \text{(all } i\text{)}, \tag{5.0.29}$$

$$\max\{s_j^-, \Sigma_{i\in S_j} \max\{0, a_i - z/w_i\}\} \leq \min\{s_j^+, \Sigma_{i\in S_j}(a_i + z/w_i)\}, \tag{5.0.30}$$

$$\Sigma_j \max\{s_j^-, \Sigma_{i\in S_j} \max\{0, a_i - z/w_i\}\} \leq 1, \tag{5.0.31}$$

$$\Sigma_j \min\{s_j^+, \Sigma_{i\in S_j}(a_i + z/w_i)\} \geq 1. \tag{5.0.32}$$

Condition (5.0.29) is satisfied for all $z \geq 0$. Relation (5.0.30) is equivalent to the pair of conditions

$$\Sigma_{i\in S_j} \max\{0, a_i - z/w_i\} \leq s_j^+,$$

$$\Sigma_{i\in S_j}(a_i + z/w_i) \geq s_j^-.$$

Finding the smallest $z \geq 0$ to satisfy the first of these relations is an instance of Problem III; the second is equivalent to

$$z \geq [s_j^- - \mathbf{a}(S_j)]/\Sigma_{i\in S_j}(1/w_i).$$

Finding the least $z \geq 0$ obeying (5.0.31) falls under Problem IX, while the same question for (5.0.32), which can be rewritten

$$\Sigma_j \max\{0, [s_j^+ - \mathbf{a}(S_j)] - z\Sigma_{i\in S_j}(1/w_i)\} \leq \Sigma_j s_j^+ - 1,$$

is an instance of Problem III.

5.1. Componentwise Bounds

As noted earlier in this section, the problems just treated are the simplest ones in the context of disaggregating an approximate distribution. We turn now to a more complex case in which $\mathbf{x}$ is also subjected to componentwise bounds, described by n-vectors $\mathbf{L}$ and $\mathbf{U}$ with

$$\mathbf{0} \leq \mathbf{L} \leq \mathbf{U} \leq \mathbf{1}, \quad \Sigma_i L_i \leq 1 \leq \Sigma_i U_i. \tag{5.1.1}$$

Thus the constraint set is given by

$$P = \{\mathbf{x}: \Sigma_i x_i = 1, \mathbf{L} \leq \mathbf{x} \leq \mathbf{U}, s_j^- \leq \mathbf{x}(S_j) \leq s_j^+ \text{ for } 1 \leq j \leq m\}. \tag{5.1.2}$$

To simplify later notation, we can without loss of generality adjust $\mathbf{s}^-$ and $\mathbf{s}^+$ so that

$$\mathbf{L}(S_j) \leq s_j^- \leq s_j^+ \leq \mathbf{U}(S_j). \tag{5.1.3}$$

The conditions for P to be nonempty are given by the analysis of Problem VIII in section 2; besides (5.1.1) and (5.1.3), they are

$$\Sigma_j s_j^- \leq 1 \leq \Sigma_j s_j^+. \tag{5.1.4}$$

We begin as before by determining the appropriate M_i and m_i. For a fixed $J\epsilon\{1,2,\ldots,m\}$, a fixed $i\epsilon S_J$, and a fixed $x_i\epsilon[L_i,U_i]$, consider Problem VIII in section 2 with $S=1$, $A_i=B_i=x_i$, and

$$A_k=L_k,\ B_k=U_k \text{ for all } k\neq i.$$

Of the conditions for a solution to exist, (2.30) is automatically satisfied, as is (2.31) for all $j\neq J$. For $j=J$, (2.31) yields

$$\max\{s_J^-,\ \mathbf{L}(S_J)+x_i-L_i\}\leq\min\{s_J^+,\ \mathbf{U}(S_J)-(U_i-x_i)\},\tag{5.1.5}$$

while (2.32) translates into

$$\max\{s_J^-,\ \mathbf{L}(S_J)+x_i-L_i\}+\Sigma_{j\neq J}s_j^-\leq 1,\tag{5.1.6}$$

$$\min\{s_J^+,\ \mathbf{U}(S_J)-(U_i-x_i)\}+\Sigma_{j\neq J}s_j^+\geq 1.\tag{5.1.7}$$

From the last three inequalities, it follows that

$$M_i=\min\{U_i,\ s_J^++L_i-\mathbf{L}(S_J),\ 1-\Sigma_{j\neq J\epsilon j}^-+L_i-\mathbf{L}(S_J)\},\tag{5.1.8}$$

$$m_i=\max\{L_i,\ s_J^-+U_i-\mathbf{U}(S_J),\ 1-\Sigma_{j\neq J}s_j^++U_i-\mathbf{U}(S_J)\}.\tag{5.1.9}$$

Relations (1.6)-(1.8) yield

$$\max\{L_i,\ M_i-z/w_i\}\leq x_i\leq\min\{U_i,m_i+z/w_i\},$$

$$\Sigma_i x_i=1,$$

$$s_j^-\leq\mathbf{x}(S_j)\leq s_j^+.$$

By the analysis of Problem VIII in section 2, this (for given z) has a solution $\mathbf{x}$ if and only if

$$\max\{L_i,\ M_i-z/w_i\}\leq\min\{U_i,\ m_i+z/w_i\},\tag{5.1.10}$$

$$\max\{s_j^-,\ \Sigma_{i\epsilon S_j}\max\{L_i,\ M_i-z/w_i\}\}\leq\min\{s_j^+,\ \Sigma_{i\epsilon S_j}\min\{U_i,\ m_i+z/w_i\}\}.\tag{5.1.11}$$

$$\Sigma_j\max\{s_j^-,\ \Sigma_{i\epsilon S_j}\max\{L_i,\ M_i-z/w_i\}\}\leq 1,\tag{5.1.12}$$

$$\Sigma_j\min\{s_j^+,\ \Sigma_{i\epsilon S_j}\min\{U_i,\ m_i+z/w_i\}\}\geq 1.\tag{5.1.13}$$

We seek the smallest value $z_{\min}$ of $z\geq 0$ which satisfies these four conditions.

Let z° be the least z satisfying (5.1.10). As in (3.1.8), the result is

$$z^\circ=\tfrac{1}{2}\max_i w_i(M_i-m_i),\tag{5.1.14}$$

where M_i-m_i can be evaluated using (5.1.8-9).

The new information in (5.1.11) is given by the pair of inequalities

$$\Sigma_{i\epsilon S_j}\max\{L_i,\ M_i-z/w_i\}\leq s_j^+,$$

$$\Sigma_{i\epsilon S_j}\min\{U_i,\ m_i+z/w_i\}\geq s_j^-,$$

which can be rewritten

$$\Sigma_{i\epsilon S_j}\max\{0,\ (M_i-L_i)-z/w_i\}\leq s_j^+-\mathbf{L}(S_j),\tag{5.1.15}$$

$$\Sigma_{i\epsilon S_j}\max\{0,\ (U_i-m_i)-z/w_i\}\leq\mathbf{U}(S_j)-s_j^-\tag{5.1.16}$$

Let z_j^* and z_j^{**} denote the smallest value of $z\geq 0$ satisfying (5.1.15) or (5.1.16) respectively, and put

$$z^*=\max_j z_j^*,\ z^{**}=\max_j z_j^{**}.$$

Then finding z_j^* and z_j^{**} are instances of Problem III in section 2, so determining z^* and z^{**} can be regarded as solved problems.

Now let z^{***} be the least $z \geq 0$ satisfying (5.1.12), which we rewrite as

$$\Sigma_j \max \{s_j^- - \mathbf{L}(S_j), \Sigma_{i \epsilon S_j} \max \{0, \ (M_i - L_i) - z/w_i\}\} \leq 1 - \Sigma_i L_i,$$

and let z^{****} be the least $z \geq 0$ satisfying (5.1.13), which we rewrite as

$$\Sigma_j \max \{\mathbf{U}(S_j) - s_j^+, \Sigma_{i \epsilon S_j} \max \{0, \ (U_i - m_i) - z/w_i\}\} \leq \Sigma_i U_i - 1.$$

Then

$$z_{\min} = \max \{z^\circ, z^*, z^{**}, z^{***}, z^{****}\},$$

where the determination of z°, z^* and z^{**} has already been discussed. But finding z^{***} and z^{****} are instances of (solved) Problem IX in section 2. The associated minimax adjustment problem is similarly solvable; we omit details.

5.2. Componentwise Rankings and Subvectors

In section 4 we used the notation $\mathbf{x}_j$ for the subvector of $\mathbf{x}$ with components $\{x_i : i \epsilon S_j\}$. Let $\sigma(j) = |S_j|$ and let $\{x_r^j\}_{r=1}^{\sigma(j)}$ be an enumeration of the components of $\mathbf{x}_j$. In this subsection we assume the "incomplete information" on $\mathbf{x}$, apart from the disaggregation conditions.

$$s_j^- \leq \mathbf{x}(S_j) \leq s_j^+ \tag{5.2.1}$$

and the ubiquitous

$$\Sigma_i x_i = 1, \tag{5.2.2}$$

to be a prescribed rank order within each subvector,

$$x_1^j \leq x_2^j \leq \ldots \leq x_{\sigma(j)}^j. \tag{5.2.3}$$

We begin by determining the M_i and m_i. If $x_i = x_r^j$, write w_r^j, M_r^j and m_r^j for w_i, M_i and m_i. The analysis proceeds in two steps.

First, assume the vector $\mathbf{s}$ with $s_j = \mathbf{x}(S_j)$ is known. Then for each $j \epsilon \{1,2,\ldots,m\}$, a rescaling by $1/s_j$ converts the problem into that leading to the previous equations (3.2.2-4). Thus, using a functional notation to express dependence on s_j, we have

$$M_r^j(s_j) = s_j/[\sigma(j) + 1 - r].$$

$$m_r^j(s_j) = 0 \qquad \text{if } r < \sigma(j),$$

$$m_{\sigma(j)}^j(s_j) = s_j/\sigma(j).$$

Second, we take into account the possible range of variation of s_j, constrained as it is by

$$s_j^- \leq s_j \leq s_j^+ \tag{5.2.4}$$

$$\Sigma_j s_j = 1. \tag{5.2.5}$$

Determining this range is precisely the problem solved earlier to yield eqs. (3.1.13-14). Thus we obtain

$$M_r^J = \min \{s_J^+, 1 - \Sigma_{j \neq J} s_j^-\}/[\sigma(J) + 1 - r], \tag{5.2.6}$$

$$m_r^J = 0 \qquad \text{if } r < \sigma(J), \tag{5.2.7}$$

$$m_{\sigma(J)}^J = \max \{s_J^-, 1 - \Sigma_{j \neq J} s_j^+\}/\sigma(J). \tag{5.2.8}$$

For the main analysis, we wish to determine the least $z \geq 0$ for which an $\mathbf{x}$ exists satisfying (5.2.1-3) as well as

$$\max \{0, M_i - z/w_i\} \leq x_i \leq m_i + z/w_i.$$

Again we proceed in two steps, first treating all $s_j = \mathbf{x}(S_j)$ as known. Thus for all j, we seek a subvector $\mathbf{x}_j$ which satisfies (5.2.3) as well as

$$\max \{0, M_r^j - z/w_r^j\} \leq x_r^j \leq m_r^j + z/w_r^j \tag{5.2.9}$$

for $1 \leq r \leq \sigma(j)$, and

$$\Sigma_{r=1}^{\sigma(j)} x_r^j = s_j. \tag{5.2.10}$$

This is an instance of Problem II in section 2; the conditions for solutions $\mathbf{x}_j$ to exist read

$$\max \{0, M_r^j - z/w_r^j\} \leq m_R^j + z/w_R^j \qquad (r \leq R) \tag{5.2.11}$$

$$\Sigma_{r=1}^{\sigma(j)} \max_{R \leq r} \{0, M_R^j - Z/w_R^j\} \leq s_j, \tag{5.2.12}$$

$$\Sigma_{r=1}^{\sigma(j)} \min_{R \geq r} (m_R^j + z/w_R^j) \geq s_j. \tag{5.2.13}$$

Here (5.2.11) does not involve s_j; the least value of z which satisfies it is readily found *via*

$$z_j^\circ = \max_{1 \leq r \leq R \leq \sigma(j)} \{(M_r^j - m_R^j)/(1/w_r^j + 1/w_R^j)\}, \tag{5.2.14}$$

$$z^\circ = \max \{0, \max_j z_j^\circ\}. \tag{5.2.15}$$

Next, rewrite (5.2.12) as

$$\Sigma_{r=1}^{\sigma(j)} \max \{0, \max_{R \leq r} (M_R^j - z/w_R^j)\} \leq s_j \tag{5.2.16}$$

and repeat (5.2.13) as

$$\Sigma_{r=1}^{\sigma(j)} \min_{R \geq r} (m_R^j + z/w_R^j) \geq s_j. \tag{5.2.17}$$

The question of existence for a vector $\mathbf{s}$ obeying both (5.2.4-5) and (5.2.16-17) is an instance of Problem I in section 2; the conditions for existence are

$$\max \{s_j^-, \Sigma_{r=1}^{\sigma(j)} \max \{0, \max_{R \leq r} (M_R^j - z/w_R^j)\}\} \leq \min \{s_j^+, \Sigma_{r=1}^{\sigma(j)} \min_{R \geq r} (m_R^j + z/w_R^j)\} \tag{5.2.18}$$

for all j, and

$$\Sigma_j \max \{s_j^-, \Sigma_{r=1}^{\sigma(j)} \max \{0, \max_{R \leq r} (M_R^j - z/w_R^j)\}\} \leq 1, \tag{5.2.19}$$

$$\Sigma_j \min \{s_j^+, \Sigma_{r=1}^{\sigma(j)} \min_{R \geq r} (m_R^j + z/w_R^j)\} \geq 1. \tag{5.2.20}$$

The new information deducible from (5.2.18) consists of the two inequalities

$$\Sigma_{r=1}^{\sigma(j)} \max \{0, \max_{R \leq r} (M_R^j - z/w_R^j)\} \leq s_j^+, \tag{5.2.21}$$

$$\Sigma_{r=1}^{\sigma(j)} \min_{R \geq r} (m_R^j + z/w_R^j) \geq s_j^-. \tag{5.2.22}$$

Finding the least $z \geq 0$ satisfying (5.2.21) is an instance of unsolved Problem V in section 2; similarly for (5.2.19) and unsolved Problem X. In view of (5.2.7), relation (5.2.22) can be rewritten

$$(m^j + z/w^j) + \Sigma_{r < \sigma(j)} \min \{m^j + z/w^j, z/v_r^j\} \geq s_j^-$$

$$(m^j = m_{\sigma(j)}^j, w^j = w_{\sigma(j)}^j, v_r^j = \max_{r \leq R < \sigma(j)} w_R^j), \tag{5.2.23}$$

and the least $z \geq 0$ satisfying it can be found using the technique illustrated near (3.2.20-23) in the previous text. Finding the least $z \geq 0$ obeying (5.2.20) is *a priori* an instance of unsolved Problem XI, and appeal to (5.2.7) has not so far yielded reduction to a solved Problem despite the initial simplification to

$$\Sigma_j \min \{s_j^+, m^j + z/w^j + \Sigma_{r<\sigma(j)} \min \{m^j + z/w^j, z/v_r^j\}\} \geq 1. \tag{5.2.24}$$

In view of these difficulties, we first specialize to the case

$$\{w_r^j\}_{r=1}^{\sigma(j)} \text{ nondecreasing, for all } j. \tag{5.2.25}$$

Then (5.2.19) and (5.2.21) become

$$\Sigma_j \max \{s_j^-, \Sigma_{r=1}^{\sigma(j)} \max \{0, M_r^j - z/w_r^j\}\} \leq 1, \tag{5.2.26}$$

$$\Sigma_{r=1}^{\sigma(j)} \max \{0, M_r^j - z/w_r^j\} \leq s_j^+, \tag{5.2.27}$$

and so can be handled as instances of (solved) Problems IX and III respectively. The treatment of (5.2.23) simplifies as in the analysis leading to (3.2.26). Equation (5.2.14), or more directly (5.2.11), gives

$$z_j^\circ = \tfrac{1}{2} w^j - (M^j - m^j) \qquad (M^j = M_{\sigma(j)}^j). \tag{5.2.28}$$

However, (5.2.24) still appears recalcitrant despite the additional information $v_r^j \leq w^j$.

We therefore retreat still further, to the semi-weighted case ($w_i = W_j$ for all $i \epsilon S_j$). Now (5.2.23) admits explicit solution

$$z \geq W_j [s_j^- - m^j] / \sigma(j).$$

Most important, the previous maverick (5.2.24) becomes equivalent to

$$\Sigma_j \max \{0, (s_j^+ - m^j) - \sigma(j) z / W_j\} \leq \Sigma_j s_j^+ - 1 \tag{5.2.29}$$

so that its treatment is an instance of (solved) Problem III.

We turn now to the minimax adjustment problem. Let $a_i = a_r^j$ if $x_i = x_r^j$, and again begin by treating $\mathbf{s}$ as if it were known. Then the constraints on the components of subvector $\mathbf{x}_j$ read

$$\max \{0, a_r^j - z/w_r^j\} \leq x_r^j \leq a_r^j + z/w_r^j,$$

$$\Sigma_{r=1}^{\sigma(j)} x_r^j = s_j,$$

$$x_1^j \leq x_2^j \leq \ldots \leq x_{\sigma(j)}^j.$$

The conditions for such an $\mathbf{x}_j$ to exist are

$$\max \{0, \max_{R<r} (a_R^j - z/w_R^j)\} \leq \min_{R \geq r} (a_R^j + z/w_R^j), \tag{5.2.30}$$

$$\Sigma_{r=1}^{\sigma(j)} \max \{0, \max_{R \leq r} (a_R^j - z/w_R^j)\} \leq s_j, \tag{5.2.31}$$

$$\Sigma_{r=1}^{\sigma(j)} \min_{R \geq r} (a_R^j + z/w_R^j) \geq s_j. \tag{5.2.32}$$

The least z satisfying (5.2.30) is given by

$$z_j^\circ = \max_{r \leq R} \{(a_r^j - a_R^j)/(1/w_r^j + 1/w_R^j)\}. \tag{5.2.33}$$

The conditions for the existence of an $\mathbf{s}$ satisfying (5.2.31-32) as well as (5.2.4-5) are

$$\max \{s_j^-, \Sigma_{r=1}^{\sigma(j)} \max \{0, \max_{R<r} (a_R^j - z/w_R^j)\}\} \leq \min \{s_j^+, \Sigma_{r=1}^{\sigma(j)} \min_{R \geq r} (a_R^j + z/w_R^j)\}, \tag{5.2.34}$$

$$\Sigma_j \max \{s_j^-, \Sigma_{r=1}^{\sigma(j)} \max \{0, \max_{R<r} (a_R^j - z/w_R^j)\}\} \leq 1, \tag{5.2.35}$$

$$\Sigma_j \min \{s_j^+, \Sigma_{r=1}^{\sigma(j)} \min_{R \geq r} (a_R{}^j + z/w_R{}^j)\} \geq 1. \tag{5.2.36}$$

The new information from (5.2.34) is given by

$$\Sigma_{r=1}^{\sigma(j)} \max \{0, \max_{R \leq r} (a_R{}^j - z/w_R{}^j)\} \leq s_j^+, \tag{5.2.37}$$

$$\Sigma_{r=1}^{\sigma(j)} \min_{R \geq r} (a_R{}^j + z/w_R{}^j) \geq s_j^-. \tag{5.2.38}$$

Here we have instances of unsolved Problems V, VI, X and XI; again we pass to the semi-weighted case $w_r{}^j = W_j$. It is convenient to define

$$b_r{}^j = \max_{R \leq r} a_R{}^j, \; c_r{}^j = \min_{R \geq r} a_R{}^j. \tag{5.2.39}$$

Then (5.2.33) yields

$$z_j{}^\circ = \tfrac{1}{2} W_j \max_r (b_r{}^j - c_r{}^j). \tag{5.2.40}$$

Relation (5.2.37) becomes

$$\Sigma_{r=1}^{\sigma(j)} \max \{0, b_r{}^j - z/W_j\} \leq s_j^+, \tag{5.2.41}$$

so determining the least value $z_j{}^*$ of $z \geq 0$ which satisfies it is an instance of Problem III; the least value of z which satisfies (5.2.38) is

$$z_j{}^{**} = [s_j^- - \Sigma_{r=1}^{\sigma(j)} c_r{}^j)] W_j/\sigma(j). \tag{5.2.42}$$

Continuing, (5.2.35) becomes

$$\Sigma_j \max \{s_j^- - \Sigma_{r=1}^{\sigma(j)} \max \{0, b_r{}^j - z/W_j\}\} \leq 1, \tag{5.2.43}$$

while (5.2.36) becomes equivalent to

$$\Sigma_j \max \{0, [s_j^+ - \Sigma_{r=1}^{\sigma(j)} c_r{}^j] - z\sigma(j)/W_j\} \leq \Sigma_j s_j^+ - 1. \tag{5.2.44}$$

Finding the least values of $z \geq 0$ which satisfy these (z^{***} and z^{****}) are instances of Problems IX and III. Thus finding the appropriate $z_{\min}$ can again be regarded as a solved problem.

5.3. Both Componentwise Bounds and Rankings

In this subsection, we impose *both* the constraints of the previous two subsections. Thus, writing $L_r{}^j$ and $U_r{}^j$ for $L_i \geq 0$ and $U_i \leq 1$ if $x_i = x_r{}^j$, our polyhedron P in $\mathbf{x}$-space is defined by

$$L_r{}^j \leq x_r{}^j \leq U_r{}^j \tag{5.3.1}$$

$$\Sigma_{r=1}^{\sigma(j)} x_r{}^j = s_j, \tag{5.3.2}$$

$$x_1{}^j \leq x_2{}^j \leq \ldots \leq x_{\sigma(j)}{}^j, \tag{5.3.3}$$

$$s_j^- \leq s_j \leq s_j^+, \tag{5.3.4}$$

$$\Sigma_j s_j = 1. \tag{5.3.5}$$

Without loss of generality, it may be assumed that

$$\{L_r{}^j\}_{r=1}^{\sigma(j)} \quad \text{and} \quad \{U_r{}^j\}_{r=1}^{\sigma(j)} \text{ are nondecreasing,} \tag{5.3.6}$$

$$\mathbf{L}(S_j) = \Sigma_r L_r{}^j \leq s_j^- \leq s_j^+ \leq \Sigma_r U_r{}^j = \mathbf{U}(S_j). \tag{5.3.7}$$

We first determine the conditions for P to be nonempty. For given $\mathbf{s}$, finding an $\mathbf{x}_j$ satisfying (5.3.1-3) is an instance of Problem II; using (5.3.6), the conditions for a solution to exist are

$$L_r^j \leq U_r^j, \tag{5.3.8}$$

$$\Sigma_r L_r^j \leq s_j \leq \Sigma_r U_r^j. \tag{5.3.9}$$

It will be assumed that (5.3.8) holds. In view of (5.3.7), (5.3.9) is already implied by (5.3.4) and so can be ignored. Thus the remaining condition for P to be nonempty is the existence of an $\mathbf{s}$ satisfying (5.3.4) and (5.3.5); this is an instance of Problem I, and the only new condition introduced is

$$\Sigma_j s_j^- \leq 1 \leq \Sigma_j s_j^+. \tag{5.3.10}$$

Assume for the moment that the M_i and m_i appropriate to this polyhedron P are known, and observe that

$$\{M_r^j\}_{r=1}^{\sigma(j)} \quad \text{and} \quad \{m_r^j\}_{r=1}^{\sigma(j)} \text{ are nondecreasing.} \tag{5.3.11}$$

We wish to determine the least $z \geq 0$ for which an $\mathbf{x}$ and $\mathbf{s}$ exist satisfying (5.3.1-5) as well as

$$\max\{0, M_r^j - z/w_r^j\} \leq x_r^j \leq m_r^j + z/w_r^j. \tag{5.3.12}$$

The conditions for the existence of an $\mathbf{x}_j$ satisfying (5.3.12) as well as (5.3.1-3) are

$$\max\{L_r^j, M_r^j - z/w_r^j\} \leq \min\{U_R^j, m_R^j + z/w_R^j\} \qquad (r \leq R), \tag{5.3.13}$$

$$\Sigma_{r=1}^{\sigma(j)} \max_{R \leq r} \max\{L_R^j, M_R^j - z/w_R^j\} \leq s_j, \tag{5.3.14}$$

$$\Sigma_{r=1}^{\sigma(j)} \min_{R \geq r} \min\{U_R^j, m_R^j + z/w_R^j\} \geq s_j. \tag{5.3.15}$$

The least $z \geq 0$ satisfying (5.3.13) is given, as in (5.2.14), by

$$z_j^\circ = \max\{0, \max_{1 \leq r \leq R \leq \sigma(j)} \{(M_r^j - m_R^j)/(1/w_r^j + 1/w_R^j)\}\}. \tag{5.3.16}$$

Next, by Problem I, an $\mathbf{s}$ satisfying (5.3.4-5) as well as (5.3.14-15) exists if and only if

$$\Sigma_{r=1}^{\sigma(j)} \max_{R \leq r} \max\{L_R^j, M_R^j - z/w_R^j\} \leq s_j^+, \tag{5.3.17}$$

$$\Sigma_{r=1}^{\sigma(j)} \min_{R \geq r} \min\{U_R^j, m_R^j + z/w_R^j\} \geq s_j^-, \tag{5.3.18}$$

$$\Sigma_j \max\{s_j^-, \Sigma_{r=1}^{\sigma(j)} \max_{R \leq r} \max\{L_R^j, M_R^j - z/w_R^j\}\} \leq 1,$$

$$\Sigma_j \min\{s_j^+, \Sigma_{r=1}^{\sigma(j)} \min_{R \geq r} \min\{U_R^j, m_R^j + z/w_R^j\}\} \geq 1.$$

The last two relations can be rewritten, using (5.3.6), as

$$\Sigma_j \max\{s_j^-, \Sigma_{r=1}^{\sigma(j)} \max\{L_r^j, \max_{R \leq r}(M_R^j - z/w_R^j)\}\} \leq 1,$$

$$\Sigma_j \min\{s_j^+, \Sigma_{r=1}^{\sigma(j)} \min\{U_r^j, \min_{R \geq r}(m_R^j + z/w_R^j)\}\} \geq 1.$$

These in turn can be rewritten

$$\Sigma_j \max\{s_j^- - \mathbf{L}(S_j), \Sigma_{r=1}^{\sigma(j)} \max\{0, \max_{R \leq r}(M_R^j - L_r^j - z/w_R^j)\}\} \leq 1 - \Sigma_i L_i, \tag{5.3.19}$$

$$\Sigma_j \max\{\mathbf{U}(S_j) - s_j^+, \Sigma_{r=1}^{\sigma(j)} \max\{0, \max_{R \geq r}(U_R^j - z/w_R^j)\}\} \leq \Sigma_i U_i - 1. \tag{5.3.20}$$

Since (5.3.17) can be written

$$\Sigma_{r=1}^{\sigma(j)} \max\{0, \max_{R \leq r}(M_R^j - L_r^j - z/w_R^j)\} \leq s_j^+ - \mathbf{L}(S_j), \tag{5.3.21}$$

finding the least $z \geq 0$ which satisfies it is an instance of (unsolved) Problem V; the same is true for (5.3.18), which can be rewritten

$$\Sigma_{r=1}^{\sigma(j)} \max \{0, \max_{R \geq r}(U_r{}^j - m_R{}^j - z/w_R{}^j)\} \leq \mathbf{U}(S_j) - s_j{}^-. \tag{5.3.22}$$

Finding the least $z \geq 0$ which satisfies (5.3.19) and (5.3.20) is an instance of (unsolved) Problem X.

In view of these difficulties we again withdraw to the semi-weighted case ($w_r{}^j = W_j$). Then (5.3.16) becomes

$$z_j{}^\circ = \tfrac{1}{2} W_j \max_r (M_r{}^j - m_r{}^j). \tag{5.3.23}$$

(5.3.21) and (5.3.22) become

$$\Sigma_{r=1}^{\sigma(j)} \max \{0, (M_r{}^j - L_r{}^j - z/W_j)\} \leq s_j{}^+ - \mathbf{L}(S_j), \tag{5.3.24}$$

$$\Sigma_{r=1}^{\sigma(j)} \max \{0, (U_r{}^j - m_r{}^j) - z/W_j\} \leq \mathbf{U}(S_j) - s_j{}^-, \tag{5.3.25}$$

so that finding the least $z \geq 0$ (call it $z_j{}^*$) satisfying (5.3.24), or the least $z \geq 0$ (call it $z_j{}^{**}$) satisfying (5.3.25), is an instance of Problem III. Similarly, (5.3.19) and (5.3.20) become

$$\Sigma_j \max \{s_j{}^- - \mathbf{L}(s_j), \Sigma_{r=1}^{\sigma(j)} \max \{0, (M_r{}^j - L_r{}^j) - z/W_j\}\} \leq 1 - \Sigma_i L_i, \tag{5.3.26}$$

$$\Sigma_j \max \{\mathbf{U}(S_j) - s_j{}^+, \Sigma_{r=1}^{\sigma(j)} \max \{0, (U_r{}^j - m_r{}^j) - z/W_j\}\} \leq \Sigma_i U_i - 1. \tag{5.3.27}$$

Finding the least $z \geq 0$ (call it z^{***}) which satisfies (5.3.26), or the least $z \geq 0$ (call it z^{****}) which satisfies (5.3.27), is an instance of Problem IX. Thus, determining the desired

$$z_{\min} = \max \{\max_j (z_j{}^\circ, z_j{}^*, z_j{}^{**}), z^{***}, z^{****}\}$$

can be regarded as a solved problem, given the M_i and m_i.

We turn now to the associated minimax adjustment problem. First, finding an $\mathbf{x}_j$ which satisfies

$$a_r{}^j - z/w_r{}^j \leq x_r{}^j \leq a_r{}^j + z/w_r{}^j$$

in addition to (5.3.1-3) is an instance of Problem II; a solution exists if and only if

$$\max \{L_r{}^j, a_r{}^j - z/w_r{}^j\} \leq \min \{U_R{}^j, a_R{}^j + z/w_R{}^j\} \qquad (r \leq R), \tag{5.3.28}$$

$$\Sigma_{r=1}^{\sigma(j)} \max_{R \leq r} \max \{L_R{}^j, a_R{}^j - z/w_R{}^j\} \leq s_j \tag{5.3.29}$$

$$\Sigma_{r=1}^{\sigma(j)} \min_{R \geq r} \min \{U_R{}^j, a_R{}^j + z/w_R{}^j\} \geq s_j. \tag{5.3.30}$$

The least $z \geq 0$ satisfying (5.3.28) is given by

$$z_j{}^\circ = \max \{0, z_j{}^{\circ\circ}, z_j{}^{\circ\circ\circ}\}, \tag{5.3.31}$$

where

$$z_j{}^{\circ\circ} = \max_{r<R} [(a_r{}^j - a_R{}^j)/(1/w_r{}^j + 1/w_R{}^j)], \tag{5.3.32}$$

$$z_j{}^{\circ\circ\circ} = \max_r w_r{}^j \max(L_r{}^j - a_r{}^j, a_r{}^j - U_r{}^j). \tag{5.3.33}$$

Next, an $\mathbf{s}$ satisfying both (5.3.4-5) and (5.3.29-30) exists if and only if

$$\left.\begin{aligned} &\Sigma_{r=1}^{\sigma(j)} \max_{R \leq r} \max \{L_R{}^j, a_R{}^j - z/w_R{}^j\} \leq s_j{}^+ \\ &\Sigma_{r=1}^{\sigma(j)} \min_{R \geq r} \min \{U_R{}^j, a_R{}^j + z/w_R{}^j\} \geq s_j{}^- \\ &\Sigma_j \max \{s_j{}^-, \Sigma_{r=1}^{\sigma(j)} \max_{R \leq r} \max \{L_R{}^j, a_R{}^j - z/w_R{}^j\}\} \leq 1, \\ &\Sigma_j \min \{s_j{}^+, \Sigma_{r=1}^{\sigma(j)} \min_{R \geq r} \min \{U_R{}^j, a_R{}^j + a/w_R{}^j\}\} \geq 1. \end{aligned}\right\} \tag{5.3.34}$$

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These four relations can be rewritten, respectively, as

$$\Sigma_{r=1}^{\sigma(j)} \max \{0, \max_{R \leq r}(a_R{}^j - L_r{}^j - z/w_R{}^j)\} \leq s_j{}^+ - \mathbf{L}(S_j), \tag{5.3.35}$$

$$\Sigma_{r=1}^{\sigma(j)} \max \{0, \max_{R \geq r}(U_r{}^j - a_R{}^j - z/w_R{}^j)\} \leq \mathbf{U}(S_j) - s_j{}^-, \tag{5.3.36}$$

$$\Sigma_j \max \{s_j{}^- - \mathbf{L}(S_j), \Sigma_{r=1}^{\sigma(j)} \max \{0, \max_{R \leq r}(a_R{}^j - L_r{}^j - z/w_R{}^j)\}\} \leq 1 - \Sigma_i L_i, \tag{5.3.37}$$

$$\Sigma_j \max \{\mathbf{U}(S_j) - s_j{}^+, \Sigma_{r=1}^{\sigma(j)} \max \{0, \max_{R \geq r}(U_r{}^j - a_R{}^j - z/w_R{}^j)\}\} \leq \Sigma_i U_i - 1. \tag{5.3.38}$$

Finding the least $z \geq 0$ which satisfies (5.3.35) or (5.3.36) is an instance of unsolved Problem V; finding the least $z \geq 0$ which satisfies (5.3.37) or (5.3.38) is an instance of unsolved Problem X. Thus we revert to the semi-weighted case. Then (5.3.31) yields

$$z_j{}^{\infty} = \tfrac{1}{2} W_j \max_r (b_r{}^j - c_r{}^j), \tag{5.3.39}$$

where $b_r{}^j$ and $c_r{}^j$ are defined in (5.2.39).

Relations (5.3.35-38) become

$$\Sigma_{r=1}^{\sigma(j)} \max \{0, (b_r{}^j - L_r{}^j) - z/W_j\} \leq s_j{}^+ - \mathbf{L}(S_j), \tag{5.3.40}$$

$$\Sigma_{r=1}^{\sigma(j)} \max \{0, (U_r{}^j - c_r{}^j) - z/W_j\} \leq \mathbf{U}(S_j) - s_j{}^-, \tag{5.3.41}$$

$$\Sigma_j \max \{s_j{}^- - \mathbf{L}(S_j), \Sigma_{r=1}^{\sigma(j)} \max \{0, (b_r{}^j - L_r{}^j) - z/W_j\}\} \leq 1 - \Sigma_i L_i, \tag{5.3.42}$$

$$\Sigma_j \max \{\mathbf{U}(S_j) - s_j{}^+, \Sigma_{r=1}^{\sigma(j)} \max \{0, U_r{}^j - c_r{}^j - z/W_j\}\} \leq \Sigma_i U_i - 1. \tag{5.3.43}$$

Finding the least value of $z \geq 0$ (called $z_j{}^*$, $z_j{}^{**}$, z^{***}, z^{****} respectively) which satisfies each of these is an instance of either Problem III or Problem IX. Thus the determination of the desired

$$z_{\min} = \max \{\max_j \{z_j{}^{\infty}, z_j{}^{\infty\infty}, z_j{}^*, z_j{}^{**}\}, z^{***}, z^{****}\}$$

can be regarded as a solved problem.

We return now to determining the M_i and m_i, i.e., the $M_r{}^j$ and $m_r{}^j$, for the polyhedron P in $\mathbf{x}$-space defined by (5.3.1-5). Consider a fixed variable $x_i = x_r{}^j$. Regarding s_j as given, we can find an $M_r{}^j(s_j)$ and $m_r{}^j(s_j)$ for the polyhedron P_j in $\mathbf{x}_j$-space defined by (5.3.1-3). Examining the analysis at the beginning of subsection 3.3, we find (cf. (3.3.13-15)) that

$$M_r{}^j(s_j) = \min \{U_r{}^j, p_r{}^j(s_j)\}, \tag{5.3.44}$$

$$m_r{}^j(s_j) = \max \{L_r{}^j, q_r{}^j(s_j)\}, \tag{5.3.45}$$

where $p_r{}^j(s_j)$ is the unique p such that

$$p + \Sigma_{R \geq r} \max \{L_R{}^j, p\} = s_j - \Sigma_{R < r} L_R{}^j \tag{5.3.46}$$

and $q_r{}^j(s_j)$ is the unique q such that

$$q + \Sigma_{R < r} \min \{U_R{}^j, q\} = s_j{}^- \Sigma_{R \geq r} U_R{}^j. \tag{5.3.47}$$

As noted in subsection 3.3, these equations can be transformed, for any given s_j, into instances of (solved) Problem IV. Clearly $p_r{}^j(s_j)$ and $q_r{}^j(s_j)$ are increasing functions of s_j; if $t_j{}^+$ and $t_j{}^-$ denote the maximum and minimum possible values of s_j, then

$$M_r{}^j = \min \{U_r{}^j, p_j{}^r(t_j{}^+)\}, \tag{5.3.48}$$

$$m_r{}^j = \max \{L_r{}^j, q_j{}^r(t_j{}^-)\}. \tag{5.3.49}$$

Finally, the determination of $t_j{}^+$ and $t_j{}^-$ is an instance of the problem leading to (3.1.13-14); the results are

$$t_j^+ = \min\{s_j^+, 1 - \Sigma_{J \neq j} s_J^-\}, \tag{5.3.50}$$

$$t_j^- = \max\{s_j^-, 1 - \Sigma_{J \neq j} s_J^+\}. \tag{5.3.51}$$

5.4. Mixed Componentwise Bounds and Rankings

In this section, we consider the case in which some components s_j of the "fuzzy" distribution $\mathbf{s}$ have their disaggregation constrained by upper and lower bounds, while the disaggregations of the remaining components are subject to ranking conditions. Specifically, there is a partition

$$\{1,2,\ldots,m\} = B \cup Q$$

into nonempty sets, and for convenience we define

$$I = \cup \{S_j : j \in B\}.$$

The relevant polyhedron P in $\mathbf{x}$-space is defined by

$$0 \leq x_1^j \leq x_2^j \leq \ldots x_{\sigma(j)}^j \qquad (j \in Q), \tag{5.4.1}$$

$$L_i \leq x_i \leq U_i \qquad (i \in I), \tag{5.4.2}$$

$$\Sigma_{i \in S_j} x_i = s_j, \tag{5.4.3}$$

$$s_j^- \leq s_j \leq s_j^+, \tag{5.4.4}$$

$$\Sigma_j s_j = 1. \tag{5.4.5}$$

It will be assumed that

$$0 \leq L_i \leq U_i \qquad \text{for } i \in I, \tag{5.4.6}$$

$$\mathbf{L}(S_j) \leq s_j^- \leq s_j^+ \leq \mathbf{U}(S_j) \qquad \text{for } j \in B, \tag{5.4.7}$$

$$\Sigma_j s_j^- \leq 1 \leq \Sigma_j s_j^+. \tag{5.4.8}$$

We first observe that for any $\mathbf{s}$ satisfying (5.4.4-5), there exists an $\mathbf{x}$ satisfying (5.4.1-3). Indeed, for each $j \in Q$ there is clearly an $\mathbf{x}_j$ obeying (5.4.1) and (5.4.3); for each $j \in B$ the existence of an $\mathbf{x}_j$ satisfying (5.4.2) and (5.4.3) is an instance of Problem I, and the associated conditions

$$L_i \leq U_i, \mathbf{L}(S_j) \leq s_j \leq \mathbf{U}(S_j)$$

are indeed satisfied, by virtue of (5.4.6-7).

Next we determine the M_i and m_i. First, for each $j \in Q$, the corresponding M_r^j and m_r^j are as in subsection 5.2 (see (5.2.6-8)); this follows from the observation above. Next, for a fixed $j \in B$, $i \in S_j$, and $x_i \in [L_i, U_i]$, consider Problem I applied to $\mathbf{x}_j$ with $S = s_j$, $A_i = B_i = x_i$, and

$$A_k = L_k, B_k = U_k \qquad \text{for } k \in S_j - \{i\}.$$

The conditions for a solution are

$$x_i + \Sigma_{k \neq i} L_k \leq s_j \leq x_i + \Sigma_{k \neq i} U_i,$$

so that (with $S(j) = S_j$)

$$M_i(s_j) = \min\{U_i, s_j - \Sigma_{k \in S(j) - \{i\}} L_k\}, \tag{5.4.9}$$

$$m_i(s_j) = \max\{L_i, s_j - \Sigma_{k \in S(j) - \{i\}} U_k\}, \tag{5.4.10}$$

for $i \in S_j$, $j \in B$. Since $M_i(s_j)$ and $m_i(s_j)$ are nondecreasing in s_j, M_i is obtained as $M_i(t_j^+)$ and m_i as $m_i(t_j^-)$

where t_j^+, t_j^- are the maximum and minimum possible values of s_j, given in (5.2.49-50). Thus

$$M_i = \min \{U_i, s_j^+ - \Sigma_{k \in S(j)-\{i\}} L_k,\ 1 - \Sigma_{J \neq j} s_j^- - \Sigma_{k \in S(j)-\{i\}} L_k\}, \tag{5.4.11}$$

$$m_i = \max \{L_i, s_j^- - \Sigma_{k \in S(j)-\{i\}} U_k,\ 1 - \Sigma_{J \neq j} s_j^+ - \Sigma_{k \in S(j)-\{i\}} U_k\}. \tag{5.4.12}$$

The minimax selection problem will now be taken up. For each $j \in B$, the conditions on $\mathbf{x}_j$ are

$$\max \{L_i, M_i - z/w_i\} \leq x_i \leq \min \{U_i, m_i + z/w_i\} \qquad (i \in S_j),$$

$$\Sigma_{i \in S(j)} x_i = s_j.$$

These conditions can be met, for $z \geq 0$, if and only if

$$z \geq z_j^\circ = \tfrac{1}{2} \max_{i \in S_j} w_i (M_i - m_i) \qquad (j \in B), \tag{5.4.13}$$

$$\Sigma_{i \in S_j} \max \{L_i, M_i - z/w_i\} \leq s_j \qquad (j \in B), \tag{5.4.14}$$

$$\Sigma_{i \in S_j} \min \{U_i, m_i + z/w_i\} \geq s_j \qquad (j \in B). \tag{5.4.15}$$

For each $j \in Q$, the conditions on $\mathbf{x}_j$ are (5.4.1), (5.4.3), and

$$\max \{0, M_r^j - z/w_r^j\} \leq x_r^j \leq m_r^j + z/w_r^j.$$

These conditions can be met, for $z \geq 0$, if and only if

$$\max \{0, M_r^j - z/w_r^j\} \leq m_R^j + z/w_R^j \qquad (r \leq R), \tag{5.4.16}$$

$$\Sigma_{r=1}^{\sigma(j)} \max_{R \leq r} \max \{0, M_R^j - z/w_R^j\} \leq s_j \qquad (j \in Q), \tag{5.4.17}$$

$$\Sigma_{r=1}^{\sigma(j)} \min_{R \geq r} (m_R^j + z/w_R^j) \geq s_j \qquad (j \in Q). \tag{5.4.18}$$

The new information from (5.4.16) is

$$z \geq z_j^\infty = \max_{r \leq R} [(M_r^j - m_R^j)/(1/w_r^j + 1/w_R^j)]. \tag{5.4.19}$$

Next, an s satisfying (5.4.4-5), (5.4.14-15) and (5.4.17-18) can exist if and only if

$$\Sigma_{i \in S_j} \max \{L_i, M_i - z/w_i\} \leq s_j^+ \qquad (j \in B), \tag{5.4.20}$$

$$\Sigma_{i \in S_j} \min \{U_i, m_i + z/w_i\} \geq s_j^- \qquad (j \in B), \tag{5.4.21}$$

$$\Sigma_{r=1}^{\sigma(j)} \max_{R \leq r} \max \{0, M_R^j - z/w_R^j\} \leq s_j^+ \qquad (j \in Q), \tag{5.4.22}$$

$$\Sigma_{r=1}^{\sigma(j)} \min_{R \geq r} (m_R^j + z/w_R^j) \geq s_j^- \qquad (j \in Q), \tag{5.4.23}$$

$$\Sigma_{j \in B} \max \{s_j^-, \Sigma_{i \in S_j} \max \{L_i, M_i - z/w_i\}\}$$

$$+ \Sigma_{j \in Q} \max \{s_j^-, \Sigma_{r=1}^{\sigma(j)} \max_{R \leq r} \max \{0, M_R^j - z/w_R^j\}\} \leq 1, \tag{5.4.24}$$

$$\Sigma_{j \in B} \min \{s_j^+, \Sigma_{i \in S_j} \min \{U_i, m_i + z/w_i\}\}$$

$$+ \Sigma_{j \in Q} \min \{s_j^+, \Sigma_{r=1}^{\sigma(j)} \min_{R \geq r} (m_R^j + z/w_R^j)\} \geq 1. \tag{5.4.25}$$

For $j \in B$, let z_j^* and z_j^{**} be the least values of $z \geq 0$ satisfying (5.4.20) and (5.4.21) respectively. Writing these relations as

$$\Sigma_{i\epsilon S_j} \max \{0,\ (M_i - L_i) - z/w_i\} \leq s_j^+ - \mathbf{L}(S_j), \tag{5.4.26}$$

$$\Sigma_{i\epsilon S_j} \max \{0,\ (U_i - m_i) - z/w_i\} \leq \mathbf{U}(S_j) - s_j^-, \tag{5.4.27}$$

we see that finding z_j^* and z_j^{**} are instances of Problem III.

For $j\epsilon R$, let z_j^* and z_j^{**} be the least values of $z \geq 0$ satisfying (5.4.22) and (5.4.23) respectively. Their determination is an instance of unsolved Problem V or VI respectively, and so we now specialize to the case

$$w_i = W_j \qquad (i\epsilon S_j, j\epsilon Q). \tag{5.4.28}$$

Then (5.4.22) yields

$$\Sigma_{r=1}^{\sigma(j)} \max \{0,\ M_r^j - z/W_j\} \leq s_j^+,$$

so that determining z_j^* is an instance of Problem III. Relation (5.4.23) yields

$$z_j^{**} = \max \{0,\ W_j[s_j^- - \Sigma_{i\epsilon S_j} m_i]/\sigma(j)\} \qquad (j\epsilon Q) \tag{5.4.29}$$

where the sum can be evaluated using (5.2.6-8). Relations (5.4.24) and (5.4.25) become equivalent to

$$\Sigma_{j\epsilon B} \max \{s_j^- - \mathbf{L}(S_j),\ \Sigma_{i\epsilon S_j} \max \{0,\ M_i - z/w_i\}\}$$
$$+ \Sigma_{j\epsilon Q} \max \{s_j^-,\ \Sigma_{i\epsilon S_j} \max \{0,\ M_i - z/W_j\}\} \leq 1 - \Sigma_{i\epsilon I} L_i, \tag{5.4.30}$$

$$\Sigma_{j\epsilon B} \max \{\mathbf{U}(S_j) - s_j^+,\ \Sigma_{i\epsilon S_j} \max \{0,\ (U_i - m_i) - z/w_i\}\}$$
$$+ \Sigma_{j\epsilon Q} \max \{0,\ (s_j^+ - \Sigma_{i\epsilon S_j} m_i) - z\sigma(j)/W_j\} \leq \Sigma_{i\epsilon I} U_i + \Sigma_{j\epsilon Q} s_j^+ - 1. \tag{5.4.31}$$

Let z^{***} and z^{****} be the least values of $z \geq 0$ satisfying (5.4.30) and (5.4.31) respectively; then determining either of them is an instance of Problem IX. Thus finding the desired

$$z_{\min} = \max \{\max_j \{z_j^\circ, z_j^{\infty}, z_j^*, z_j^{**}\}, z^{***}, z^{****}\}$$

can be regarded as a solved problem.

The analysis of the associated minimax adjustment problem is analogous and will not be worked out here.

6. Concluding Remarks

The following table summarizes the status of the various topics treated. The material covered omits that of subsection 5.4 (which does not fall into the general pattern) and that of subsection 5.0 (subsumed by subsection 5.1). The status of the minimax selection and minimax adjustment problems is essentially the same in every case, so the two are not considered separately. "Ordinary" refers to the analyses in section 3, "disaggregation" to the topics in sections 4 and 5. "Bottleneck" refers to the unsolved Problem (or Problems) in section 2 which stands in the way of a satisfactory solution method for the general weighted problem.

Situation	Solved for	Bottleneck
Ordinary, bounds	Weighted	
Ordinary, ranking	Unweighted*	V, VI
Ordinary, both	Unweighted	VII
Disaggregation, bounds	Weighted	
Disaggregation, ranking	Semiweighted	V, VI, X, XI
Disaggregation, both	Semiweighted	V, X

*The selection problem is also solved for $w_* = \max_i w_i$.

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Three Results for Trees, Using Mathematical Induction

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The paper illustrates the use of mathematical induction to extend results which are true for a line segment to trees. Three separate theorems are stated and proved, each of which has some importance in its own right.

Key words: Helly's theorem; mathematical induction; minimum-length coverings; node-covering; trees.

1. Introduction

This paper presents three theorems each of which extends to trees a result that is known or obvious for a line segment. While the theorems are of interest in themselves, it is the purpose of the paper to use them as examples of how the inductive process may be effective in proving theorems for trees or for graphs with treelike structures. Other examples of the application of induction to trees include the proof that a tree with n nodes has $n-1$ arcs (cf. [5],[1] p. 35), or a paper such as [4].

The definition of a tree which will be used throughout is the following:

A *tree* is a connected graph containing no cycles.

2. A Minimal Node-Cover Problem for Trees

This section deals with the following problem: Given a (finite) tree T and a collection $\{T_i\}_1^m$ of subtrees, find a set of nodes P in T of least cardinality, such that

$$P \cap T_i \neq \phi$$

for all i. This problem has been considered for more complex graphs than trees (cf. [1], for example), but the following recursion algorithm, applicable to trees, is particularly simple to use.

ALGORITHM. Consider any "tip" node (node of order 1), $x \epsilon T$. Let A_x consist of x and its associated arc.

1. If there exists some $T_j = \{x\}$, take x to be in P and derive the remaining members of P as the set of nodes which solve the problem for the tree $T' = T - A_x$, with the collection of subtrees

$$\{T_i : i = 1,2,\ldots,m, x \notin T_i\}.$$

2. If no $T_i = \{x\}$, find P as a solution to the problem on the tree T' with the collection of subtrees $\{T_i'\}_1^m$, where

$$T_i' = T_i - A_x$$

for each i.

AMS Subject Classification: 0540.

[1] Figures in brackets indicate the literature references at the end of this paper.

THEOREM 2.1. *The above algorithm produces a minimal set* P.

PROOF. Consider first case 1, where some $T_j = \{x\}$. Let P' be some solution to the reduced problem stated in the algorithm. If P is any solution to the original problem, then clearly $x \epsilon P$, while $P - \{x\}$ meets at least every T_i not containing x. Thus by the optimality of P and P', $P - \{x\}$ and P' both have the same cardinality. Clearly $\{x\} \cup P'$ is a feasible solution for the original problem.

In the second case, if $x \epsilon T_i$ for some i, then each such T_i must contain some element besides x. Since each such T_i is connected, it must therefore contain y, the node at the other end of the arc containing x, since y cuts A_x from the rest of the tree T.

Now let P' be some optimal solution set of nodes for the reduced problem of case 2. Since $T_i' \cap P' \neq \phi$ for each i, $T_i \cap P' \neq \phi$ also, so that P' is feasible for the original problem. Let P be any optimal node set for the original problem. If $x \notin P$, then P is a feasible solution for the reduced problem, so that $|P| \geq |P'|$, proving the optimality of P'. On the other hand, if $x \epsilon P$, consider the new node set

$$P_1 = P - \{x\} \cup \{y\}.$$

Since $P \cap T_i \neq \phi$ for each i, it follows that $P_1 \cap T_i \neq \phi$ for each i, since if $x \epsilon T_i$ then $y \epsilon T_i$ for any i. But again P_1 is a feasible solution for the reduced problem, so that

$$|P| = |P_1| \geq |P'|,$$

proving the optimality of P'.

3. Minimum-Length Coverings by Intersecting Subgraphs

This section deals with an extension of a problem previously considered by the author [2].

Let G be an undirected, connected graph whose edges have nonnegative lengths. If A and B are disjoint subgraphs of G, we shall use the notation $P(A, B)$ to refer to a (any) path in G with one endpoint in A, the other in B, and no other point in $A \cup B$.

Let $\{S_i\}_1^n$ be a collection of connected nonempty subgraphs of G with the property that if $S_i \cap S_{i+1} = \phi$, for any i, then there exists a unique path $P(S_i, S_{i+1})$. A collection $\{T_i\}_1^n$ of connected subgraphs of G will be called *feasible*, relative to $\{S_i\}_1^n$, if

$$S_i \subset T_i \qquad i = 1,2,\ldots,n, \tag{3.1}$$

$$T_i \cap T_{i+1} \neq \phi \qquad i = 1,2,\ldots,n-1. \tag{3.2}$$

We treat the problem of choosing among feasible $\{T_i\}_1^n$ so as to

$$\text{minimize} \sum_{i=1}^{n} |T_i|, \tag{3.3}$$

where $|T_i|$ is the sum of the lengths of all edges in T_i. A simple algorithm solves this problem. (The problem considered in [2] was the same as that treated here, except that compact intervals on the real line replaced the subgraphs T_i, above.)

ALGORITHM.

1. Let $T_1 = S_1$.
2. Having determined T_k, find T_{k+1} as follows.

   (a) If $S_{k+1} \cap T_k \neq \phi$, set $T_{k+1} = S_{k+1}$.

   (b) If $S_{k+1} \cap T_k = \phi$, let $P(S_{k+1}, T_k)$ be any path between S_{k+1} and T_k. Set $T_{k+1} = S_{k+1} \cup P(S_{k+1}, T_k)$.

THEOREM 3.1. *The algorithm gives a minimal value for* $\Sigma|T_i|$.

PROOF. The proof is by induction on the number n of subgraphs S_i. Clearly the algorithm works for one subgraph. Assuming it works for $n-1$ subgraphs, we prove its validity for n subgraphs.

First we show that there exists an optimal feasible solution $\{T_i\}_1^n$, which minimizes $\Sigma|T_i|$, for which $T_1 = S_1$. For if $\{T_i'\}_1^n$ is optimal, but $T_1' \neq S_1$, let $P(S_1,T_2')$ be any shortest path from S_1 to T_2'. Define a new collection of subgraphs

$$T_1 = S_1, \tag{3.4}$$

$$T_2 = T_2' \cup P(S_1,T_2'), \tag{3.5}$$

$$T_i = T_i' \qquad (3 \leq i \leq n). \tag{3.6}$$

Then it is clear that $S_i \subset T_i$, $1 \leq i \leq n$, and that T_2 is connected, since $P(S_1,T_2')$ is connected and intersects T_2'. Also, $T_i \cap T_{i+1} \neq \phi$, for all i, since $T_1 \cap T_2$ contains the common point on $P(S_1,T_2')$, and $T_i \cap T_{i+1} \subset T_i' \cap T'_{i+1}$ for $2 \leq i \leq n$. Next we note that

$$|P(S_1,T_2')| \leq |T_1' - S_1|, \tag{3.7}$$

for the connectivity of T_1', together with the fact that $T_1' \cap T_2' \neq \phi$, imply that there exists a path from S_1 to T_2' which is contained in T_1' but does not contain any arcs of S_1. Thus we have from (3.4) and (3.5),

$$\begin{aligned} |T_1| + |T_2| &\leq |S_1| + |T_2'| + |P(S_1,T_2')| \\ &\leq |S_1| + |T_2'| + |T_1' - S_1| \\ &= |T_1'| + |T_2'|, \end{aligned} \tag{3.8}$$

so that

$$\sum_{i=1}^{n} |T_i| \leq \sum_{i=1}^{n} |T_i'| \tag{3.9}$$

and the collection $\{T_i\}$ is optimal.

Now let $\{T_i\}_1^n$ represent any optimal collection for which $T_1 = S_1$, and let $\{T_i^*\}_1^n$ be a collection obtained from using the algorithm. We first note that $T_2^* \subset T_2$. For if $S_1 \cap S_2 \neq \phi$, this is clear, since $T_2^* = S_2$. If $S_1 \cap S_2 = \phi$, and $P(S_1, S_2)$ is the unique path from S_1 to S_2 in G, then $P(S_1, S_2) \subset T_2$, since if this were not the case then T_2 would be connected and contain S_2, while intersecting S_1, so that there would exist another path in T_2 from S_2 to S_1, contradicting the uniqueness of $P(S_1, S_2)$. Since $T_2^* = S_2 \cup P(S_1, S_2)$, by definition, $T_2^* \subset T_2$.

Thus the members of the collection $\{T_i\}_2^n$ cover the respective members of the collection $\{T_2^*, S_3, S_4, \ldots, S_n\}$ in G. If there exists a unique path in G between T_2^* and S_3 (or if $T_2^* \cap S_3 \neq \phi$), then the induction hypothesis applies to the set $\{T_i^*\}_2^n$, which was obtained by using the algorithm on $\{T_2^*, S_3, S_4, \ldots, S_n\}$, a set of $n-1$ elements, so that we have

$$\sum_{i=2}^{n} |T_i^*| \leq \sum_{i=2}^{n} |T_i|. \tag{3.10}$$

Since $|T_1| = |T_1^*|$, the proof is complete once it is shown that any path from T_2^* to S_3 in G is unique.

Assuming $T_2^* \cap S_3 = \phi$, let P and Q be two different paths from T_2^* to S_3 in G. Since $S_2 \subset T_2^*$, there exist paths M and N (possibly the same path) from S_2 to the points (possibly the same point) where P and Q, respectively, meet T_2^*. But then $M \cup P$ and $N \cup Q$ are different paths from S_2 to S_3, contrary to the initial assumption.

COROLLARY 3.2. *The algorithm is valid when* G *is a tree and the* S_i *are subtrees of* G.

PROOF. We need only show that there exists a unique path between any two disjoint subtrees

in a given tree. Suppose that there exist two distinct paths P and Q between subtrees A and B, having respective endpoints p_A and q_A in A and p_B and q_B in B. If $p_A = q_A$ and $p_B = q_B$, then $P = Q$, since there is a unique path between points in a tree. Therefore, assume $p_A \neq q_A$, and let R_A be a path between p_A and q_A in A. If $P \cap Q \neq \phi$, then clearly $P \cup Q \cup R_A$ contains a cycle. If $P \cap Q = \phi$, let R_B be a path between p_B and q_B in B. Then $P \cup Q \cup R_A \cup R_B$ is a cycle. Thus there exists a unique path between A and B.

4. Helly's Theorem for Trees

Helly's theorem [2] states, in one dimension, that a collection of closed connected segments on a line E^1 has a nonempty intersection if and only if every pair of segments has a nonempty intersection. In this section we extend the result to trees.

THEOREM 4.1. *A collection of subtrees of a tree has at least one common node if and only if every pair of subtrees has at least one common node.*

PROOF. The proof is by induction on the number of nodes. For a tree with 1 or 2 nodes the theorem is obvious.

Assume the theorem true for a tree with $n - 1$ or fewer nodes and consider a tree T with n nodes. Let $S_1, S_2, \ldots, S_m$ be a collection of subtrees of T such that

$$S_i \cap S_j \neq \phi$$

for every pair (i, j).

Let x be any node of T which is also a cut point of T, and let $C_1, C_2, \ldots, C_k$ be the connected components of $T - \{x\}$. Let

$$A = \{x\} \cup C_1,$$

$$B = \{x\} \cup \bigcup_{i=2}^{k} C_i.$$

Then A and B are subtrees of T. Furthermore, either

$$S_i \cap A \neq \phi, \quad \text{all } i,$$

or

$$S_i \cap B \neq \phi, \quad \text{all } i.$$

For if this were not the case, then we would have some $S_i \subset A - \{x\}$ and some $S_j \subset B - \{x\}$, so that $S_i \cap S_j = \phi$.

Suppose, without prejudice to the argument, that

$$S_i \cap A \neq \phi, \quad \text{all } i,$$

Let

$$S_i' = S_i \cap A$$

for each i. Then

$$S_i' \cap S_j' \neq \phi,$$

for every pair (i, j). For if $S_i' \cap S_j' = \phi$ for some (i, j), then since $S_i \cap S_j \neq \phi$ we must have a point $y \in B - \{x\}$ such that

$$y \in S_i \cap S_j.$$

But since x is a cut point and $S_i \cap A \neq \phi$, $S_j \cap A \neq \phi$, we must have $x \in S_i \cap S_j$, by the connectedness of S_i and S_j.

Thus $\{S_i'\}$ is a collection of pairwise intersecting subtrees of the finite tree A of order less than n. By the induction hypothesis, there exists a point

$$z \in \bigcap_{i=1}^{m} S_i' \subset \bigcap_{i=1}^{m} S_i ,$$

proving the theorem.

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Abstract Groups as Doubly Transitive Permutation Groups *

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(December 12, 1971)

The question considered is this: Which abstract groups have representations as doubly transitive permutation groups? Moreover, given an abstract group, can all doubly transitive representations be found? The paper is expository. Various results which bear on the question are presented in an elementary way.

Key words: Character; cosets; matrix representation; nilpotent group; normalizer; solvable group.

1. Introduction

For a positive integer, n, let S_n denote the group of all permutations on $\{1,2,\ldots,n\}$. When $n=4$, we mean by (1 2 3) the permutation which sends 1 to 2, 2 to 3, 3 to 1, and 4 to 4. The symbol e will be used to denote the identity element of any group.

Let $G=\{e, (12), (34), (12)(34)\}$ and $H=\{e, (12)(34), (13)(24), (14)(23)\}$. Then G and H are isomorphic subgroups of S_4. But, as a group of functions on $\{1,2,3,4\}$, H acts transitively while G does not.

This simple example shows that transitivity, like most of the interesting properties associated with permutation groups, is not a group theoretic property; it is not preserved under group isomorphism. We are interested in knowing the extent to which a given combinatorial property of permutation groups depends on group theoretic properties. Said another way, suppose we are given a property P of permutation groups. (1) What abstract groups can be realized as permutation groups having property P? And, (2) given an abstract group G and a property P, can we find all possible realizations of G as a permutation group possessing property P?

When P is the property of transitivity, we can answer question (1). Cayley showed that every group is isomorphic to a permutation group. His proof amounted to a construction of the regular representation of an arbitrary group. Since the regular representation is transitive, *every group can be realized as a transitive permutation group.*

In this expository paper, we are interested in letting P be the property of double transitivity. Most of the results we will develop are old. Our purpose is to bring them together for an attack on questions (1) and (2).

Of course, doubly transitive groups are transitive. We shall begin by answering question (2) when $P=$ transitivity.

2. Transitive Representations

Let G be an abstract group with subgroup H. Let $\mathscr{H}=\{x_1H, x_2H, \ldots, x_mH\}$ be the set of distinct left cosets of H in G, where $m=[G:H]$. Then G acts on $\mathscr{H}$ (written $G:\mathscr{H}$) as follows: If $g \in G$ then

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$g(x_iH) = gx_iH$ for $1 \leq i \leq m$. Clearly, G permutes the elements of $\mathscr{H}$ among themselves. Thus, G acts as a permutation group on the elements of $\mathscr{H}$. Observe that $x_jx_i^{-1}(x_iH) = x_jH$ so that $G{:}\mathscr{H}$ transitively. We say that this permutation representation of G **corresponds** to the subgroup H.

Suppose now that G is a permutation group acting transitively on $A = \{a = a_1, a_2, \ldots, a_m\}$. Let H be the subgroup of G which fixes the element a, i.e., $g \in G$ is in H if and only if $g(a) = a$. (We will sometimes write G_a for this subgroup.) Let $x_2, x_3, \ldots, x_m$ be elements of G such that $x_ia_1 = a_i, 2 \leq i \leq m$. Then it is easily verified that $\mathscr{H} = \{H, x_2H, \ldots, x_mH\}$ is the set of distinct left cosets of H in G. (Indeed, $m = [G{:}G_a]$ if and only if G acts transitively on A.)

We have established a one-to-one correspondence between the elements of A and the elements of $\mathscr{H}$. It is easily verified that the action of G on A is carried over by this correspondence to the action of G on $\mathscr{H}$ described previously. It follows that *every transitive permutation representation of an abstract group* G *corresponds to some subgroup* H. We remark that the representation of G corresponding to $\{e\}$ is the regular representation.

DEFINITION. Let H and K be subgroups of G. We say $G{:}\mathscr{H}$ and $G{:}\mathscr{K}$ in *essentially the same way* if there exists a one-to-one function f from $\mathscr{H}$ onto $\mathscr{K}$ such that $fg(a) = gf(a)$ for all $g \in G$ and $a \in \mathscr{H}$.

It is easily seen that $G{:}\mathscr{H}$ and $G{:}\mathscr{K}$ in essentially the same way if and only if H is conjugate to K in G.

We now consider the faithfulness of the representation $G{:}\mathscr{H}$. Let $g \in G$ be such that $gxH = xH$ for every $x \in G$. Then $x^{-1}gx \in H$, or $g \in xHx^{-1}$ for every $x \in G$. Thus, the kernel of the representation $G \to G{:}\mathscr{H}$ is $K = \bigcap_{x \in G} xHx^{-1}$. Clearly, K is normal in G. Of the subgroups of H which are normal in G, K is the unique maximal one. In particular, the representation $G \to G{:}\mathscr{H}$ is faithful if and only if H contains no subgroup $K \neq \{e\}$ which is normal in G. It follows immediately that if G is abelian then $G \to G{:}\mathscr{H}$ is faithful if and only if $H = \{e\}$, i.e., *the regular representation is the only faithful transitive permutation representation of an abelian group.*

3. Doubly Transitive Representations

We begin this section by disposing of a troublesome exceptional case. Suppose G has a subgroup H of index 2. It is a standard exercise to show that H is normal. Thus, the representation of G corresponding to H is of **degree** 2 (i.e., it is subgroup of S_2) and the representation group is of order 2. Therefore, the action of G on $\mathscr{H}$ must be doubly transitive.

Let G be a doubly transitive permutation group. Suppose $G{:}A$, i.e., G is a group of one-to-one functions from A onto A. As before, let G_a be the subgroup of G which fixes $a \in A$. Since G is doubly transitive on A, G_a must be transitive on $A \setminus \{a\}$. Conversely, suppose G_a is transitive on $A \setminus \{a\}$ for all $a \in A$. (We rule out the case that the cardinality of A, $|A|$, is 1.) Given two pairs of elements from A, (a_1, a_2) and (b_1, b_2) such that $a_1 \neq a_2$ and $b_1 \neq b_2$, we seek a $g \in G$ such that $g(a_1) = b_1$ and $g(a_2) = b_2$.

If there exists an $a \in A$ which is neither a_1 nor b_1 then by hypothesis, there exists a $g_1 \in G_a$ such that $g_1(a_1) = b_1$. Now, since $a_2 \neq a_1$, it follows that $g_1(a_2) \neq b_1$. But, also, $b_2 \neq b_1$. Hence, there is a $g_2 \in G_{b_1}$ such that $g_2(g_1(a_2)) = b_2$. It follows that we may take $g = g_2g_1$.

But, if $|A| \geq 3$, such an a always exists. (If $|A| = 2$, it may not, e.g., $G = \{e\}$ and $|A| = 2$ gives rise to the unpleasant situation in which G_a is transitive on $A \setminus \{a\}$ for all $a \in A$ but G is not doubly transitive.) We have proved our first result.

THEOREM 1: *Suppose* G:A. *If* $|A| \geq 3$ *then* G:A *doubly transitively if and only if* G_a *acts transitively on* $A \setminus \{a\}$ *for every* $a \in A$.

If we are willing to assume in the previous argument that $G{:}A$ transitively, we may simply choose $g_1 \in G$ such that $g_1(a_1) = b_1$. Thus, we have another version of theorem 1.

THEOREM 1′: *Assume* G:A *transitively. If* $|A| \geq 2$, *then* G:A *doubly transitively if and only if* G_a *acts transitively on* $A \setminus \{a\}$ *for any* $a \in A$.

Indeed, it is easy to see that transitivity implies that $G_a{:}A \setminus \{a\}$ transitively for any $a \in A$ if

and only if it acts transitively for every $a \in A$. Theorem 1′ is the version most often encountered in books, e.g., Passman [7, p. 16],[1] Ledermann [5, p. 85], and Wielandt [9, p. 19].

COROLLARY 1: ([1, p. 177], [5, p. 85], [9, p. 20]). *Suppose* G:A *doubly transitively. Let* |A| = m. *Then* m(m − 1) *divides* |G|.

PROOF: We have seen in section 2 that $m = [G:G_a]$ for any $a \in A$. By Theorem 1′, $m - 1 = [G_a:G_a \cap G_b]$ for any distinct pair $a, b \in A$. Hence, $[G:G_a \cap G_b] = [G:G_a][G_a:G_a \cap G_b] = m(m-1)$. Q.E.D.

Now, if G is an abstract group with a doubly transitive representation $G:\mathscr{H}$ corresponding to H, then $m = [G:H]$ and $m(m-1)$ divides $[G:K]$ where K is the kernel of the representation. Hence, certainly, $m(m-1)$ divides $|G|$. *In particular,* |G| *must be even.*

One might ask if there are doubly transitive groups $G:A$ where $|A| = m =$ the degree of G and $|G| = m(m-1)$. Burnside [1, §140] has shown that such a group exists if and only if m is a power of a prime.

THEOREM 2: *If* G:$\mathscr{H}$ *doubly transitively then* H = N(H), *the normalizer of* H *in* G, *or* [G:H] = 2.

PROOF: Suppose $g \in N(H) \setminus H$. Then $gH \in \mathscr{H}$, and $gH \neq H$. Now, $H \subset G$ is the stabilizer of $H \in \mathscr{H}$, i.e., $xH = H$ if and only if $x \in H$. By Theorem 1′, H must act transitively on $\mathscr{H} \setminus \{H\}$. But since $g \in N(H)$, we have that $hgH = gH$ for all $h \in H$. We have reached a contradiction unless $[G:H] = 2$, in which case $\mathscr{H} \setminus \{H\} = \{gH\}$.

COROLLARY 2: *Suppose* G:$\mathscr{H}$ *doubly transitively. If* G *is nilpotent then* [G:H] = 2.

PROOF: By Hall [3, Theorems 4.3.3 and 10.3.4] we may make the following definition: G is **nilpotent** if no proper subgroup of G is its own normalizer in G. The result follows from Theorem 2.

(In spite of Corollary 1 ruling out groups of odd order and Corollary 2 eliminating nilpotent groups except in the trivial case, there are solvable doubly transitive groups. The group of even permutations on four letters, A_4, is solvable.)

COROLLARY 3: *If the center of* G, Z(G), *is not the identity alone, then* G *has a faithful representation as a doubly transitive permutation group if and only if* |G| = 2.

Let H be a subgroup of G. If $Z(G) \cap H \neq \{e\}$ then the representation corresponding to H cannot be faithful. If $Z(G) \not\subset H$ then $Z(G) \setminus H \subset N(H) \setminus H$. Thus $H \neq N(H)$.

THEOREM 3: *Suppose* G:$\mathscr{H}$ *doubly transitively. Then* H *is a maximal subgroup of* G.

PROOF: Let K be the kernel of the representation $G \to G:\mathscr{H}$. Via the usual isomorphism, $(G/K):\mathscr{H}$ doubly transitively. By [7, p. 18] $(G/K):\mathscr{H}$, being doubly transitive, is primitive. By [7, p. 15] we may take as our definition of primitivity that H/K is maximal in G/K. Thus H is maximal in G.

(If G is to have a faithful representation as a permutation group on m letters then $|G|$ must divide m, the order of S_m. In particular, if the representation corresponds to H then $|G|$ divides $[G:H]!$. By the previous theorem, when $G:\mathscr{H}$ doubly transitively, $[G:H]$ is likely to be "small" compared with $|G|$.)

We are now going to use Theorem 1′ together with the discussion of §2 to obtain some necessary and sufficient conditions that the representation of G corresponding to H be doubly transitive. Again, take $\mathscr{H} = \{x_1H, \ldots, x_mH\}$, but now assume that $x_1 \in H$. To prove that $G:\mathscr{H}$ doubly transitively, we need only show the existence of $h_2, \ldots, h_m \in H$ such that $h_ix_2H = x_iH$, $2 \leq i \leq m$. Hence, we have the following result.

THEOREM 4: [7, p. 17]. *The representation of* G *corresponding to* H *is doubly transitive if and only if* G ∖ H = HgH *for any / every* g ∈ G ∖ H.

COROLLARY 4: *The representation of* G *corresponding to* H *is doubly transitive if and only if* $[G:H] = [H:g^{-1}Hg \cap H] + 1$ *for any / every* g ∈ G ∖ H.

PROOF: According to Hall [3, Theorem 1.7.1], the number of left cosets in the "double coset" HgH is $[H:g^{-1}Hg \cap H]$. According to Theorem 4, $G:\mathscr{H}$ doubly transitively if and only if this number is $[G:H] - 1$.

(Corollary 4 also follows from the proof of corollary 1.)

One immediate consequence of Corollary 4 is that $([G:H] - 1)$ divides the order of H.

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[1] Figures in brackets indicate the literature references at the end of this paper.

Before we proceed to the next result it will be convenient to define a **proper coset** of H in G as a coset of H in G which is not a subgroup of G, i.e., xH is a proper coset if and only if $x \notin H$.

COROLLARY 5: *The representation of* G *corresponding to* H *is doubly transitive if and only if for any / every* $g \in G \setminus H$, *a system of representatives for the proper left cosets of* H *in* G *may be chosen from the right coset* Hg.

PROOF: Suppose $G:\mathscr{H}$ doubly transitively. Let $g \in G \setminus H$. By Theorem 4, $HgH = G \setminus H$. Thus, there exist $e = h_2, h_3, \ldots, h_m \in H$ such that $\mathscr{H} = \{H, h_2gH, h_3gH, \ldots, h_mgH\}$. But, $h_ig \in Hg$, $2 \leq i \leq m$.

Conversely, suppose $\mathscr{H} = \{H, x_2H, \ldots, x_mH\}$ and $x_2, \ldots, x_m \in Hx$ for some $x \in G \setminus H$. Then $Hx_2 = \ldots = Hx_m$. Let $x_ix_2^{-1} = h_i \in H$, for $2 \leq i \leq m$. Then $x_i = h_ix_2$ and $x_iH = h_ix_2H$. Thus $x_iH \subset Hx_2H$, for $2 \leq i \leq m$. The result follows from Theorem 4.

It is interesting to compare Corollary 5 with the following combinatorial result:

THEOREM [8, p. 51]. *Let* G *be a finite group and let* H *be a subgroup of* G. *There exist elements* $g_1, g_2, \ldots, g_m$ *in* G *such that*

$$G = Hg_1 \cup \ldots \cup Hg_m = g_1H \cup \ldots \cup g_mH,$$

where $m = [G:H]$.

COROLLARY 6. *If the commutator subgroup of* G *is contained in* H *and if* $[G:H] > 2$, *then the representation of* G *corresponding to* H *cannot be doubly transitive.*

PROOF. Suppose C_1 and C_2 are two different proper left cosets of H in G. Suppose $G:\mathscr{H}$ doubly transitively. By Corollary 5, there exist $x, y \in G$ such that $xH = C_1$, $yH = C_2$, and $xy^{-1} \in H$. If the commutator $z = yx^{-1}y^{-1}x \in H$ then $y^{-1}x \in xy^{-1}H = H$, which is impossible since $xH \neq yH$.

4. Matrix Representations

There are many ways of expressing the elements of a permutation group. Let $A = \{1,2,\ldots,m\}$ and suppose $G:A$. We wish to represent $g \in G$ by the m-square matrix $P(g)$ whose i,j entry is 1 if $i = g(j)$ and 0 otherwise. It is easy to see that $P(e) = I$, the m-square identity matrix, and $P(g_1g_2) = P(g_1)P(g_2)$ for all $g_1, g_2 \in G$. It follows that $g \to P(g)$ in a representation in the ordinary sense of group representation theory. (If we let e_i be the $1 \times m$ column vector with a 1 in row i and 0 elsewhere, $\{P(g):g \in G\}:\{e_i:1 \leq i \leq m\}$ is "permutation isomorphic" to $G:A$.)

Suppose $G:A$. For $g \in G$, let $\theta(g)$ be the number of points (elements) of A fixed by g. Then $\theta(g)$ is the cardinality of $\{a \in A:g(a) = a\}$. Clearly $\theta(g)$ is the trace of $P(g)$. Thus θ is a character on G.

Before we can prove our next theorem, we need to do some preliminary work. Let $a, b \in A$. We say $\mathbf{a \equiv b \pmod{G}}$ if there is a $g \in G$ such that $g(a) = b$. Because of the group properties, $\equiv \pmod{G}$ is easily seen to be an equivalence relation. The equivalence classes in A induced by $\equiv \pmod{G}$ are called **orbits** of G. In particular, $G:A$ transitively if and only if A is the only orbit of G. If $O \subseteq A$ is an orbit of G, then it makes sense to write $G:O$. Of course, $G:O$ transitively.

LEMMA ([7, p. 13], [1, p. 191], [9, Exercise 3.10]). *Let* t *be the number of orbits of* G *in* A. *Then*

$$\sum_{g \in G} \theta(g) = t|G|.$$

PROOF: We count the set $S = \{(g,a):a \in A \text{ and } g \in G_a\}$ in two ways:

$$\sum_{g \in G} \theta(g) = \sum_{a \in A} |G_a|.$$

It is an easy exercise to verify that $[G:G_a]$ is the number of elements in the orbit of G to which a belongs. Thus

$$\sum_{a \in A} |G_a| = t|G| \tag{1}$$

THEOREM 5: *Suppose* G:A *where* $|A| \geq 2$. *Then*

$$|G|^{-1} \sum_{g \in G} \theta(g)^2 \geq 2$$

with equality if and only if G:A *doubly transitively.*

(Most books, e.g., [1, p. 191], [3, Theorem 16.6.14], and [7, Proposition 3.9], hypothesize that G be transitive in Theorem 5. As we shall see, this is unnecessary.)

PROOF: Following Passman [7, p. 18], we count the set $S = \{(g, a, b): a, b, \in A, g \in G_a \cap G_b\}$ in two ways. For fixed $g \in G$, there are $\theta(g)$ possibilities for a and $\theta(g)$ possibilities for b (we may have $a = b$). Thus $|S| = \Sigma\theta(g)^2$. For fixed b, there are $\theta(g)$ possibilities for a where g is allowed to range over G_b. Thus

$$|S| = \sum_{b \in A} \sum_{g \in G_b} \theta(g)$$

$$= \sum_{b \in A} t_b |G_b|,$$

by the lemma, where t_b is the number of orbits of G_b acting on A. Since G_b fixes b and $|A| \geq 2$, $t_b \geq 2$. Moreover, $t_b = 2$ if and only if G_b is transitive on $A \setminus \{b\}$. Thus

$$|S| \geq 2 \sum_{b \in A} |G_b| \tag{2}$$

with equality if and only if G_b is transitive on $A \setminus \{b\}$ for all $b \in A$. But, from (1),

$$\sum_{b \in A} |G_b| = t|G|$$

where t is the number of orbits of G in A. Hence, $|S| \geq 2t|G| \geq 2|G|$.

Suppose $|S| = 2|G|$. Then $t = 1$ meaning that G is transitive. But, we must also have equality in (2). Using Theorem 1′ we conclude that $|S| = 2|G|$ if and only if G is doubly transitive. Q.E.D.

Since θ is a real character, one recognizes that the left hand side of the inequality in Theorem 5 is, by the orthogonality relations for characters, the sum of the squares of the multiplicities of the irreducible characters belonging to θ. We have proved our next result.

COROLLARY 7 ([2, Theorem 32.5], [3, Theorem 16.6.15]): *Let* G : A. *The representation* g→ P(g) *reduces into at least two irreducible pieces, one of which is the identically* 1 *representation. It reduces into exactly two* pieces *if and only if* G:A *doubly transitively.*

Thus, if G is an abstract group with a faithful representation as a doubly transitive group, G has a faithful irreducible representation. This brings us to the following question: What abstract groups, G, have faithful irreducible representations? It is not hard to show that G must have a cyclic center. But, there exist groups having trivial center which do not have faithful irreducible representations. Burnside [1, p. 476] gives such a group of order 18. He also gives a sufficient condition that G have a faithful irreducible representation. Weisner [6, p. 68] has given a necessary and sufficient condition that G have a faithful irreducible representation. Other work on this question has been done by Kochendörffer [4].

Now, suppose one has an irreducible representation, $g \to R(g)$ of an abstract group G. When does there exist a nonsingular matrix U such that

$$U^{-1} \begin{bmatrix} 1 & 0 \\ 0 & R(g) \end{bmatrix} U = P(g) \tag{3}$$

is a permutation matrix for every $g \in G$? Of course, if such a U exists then $g \to P(g)$ is a doubly transitive representation of G. Certainly it is necessary that trace $R(g)$ be an integer ≥ -1 for every

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g. Also, except when G has a subgroup of index 2, trace $R(e)$ must be greater than 1. But, these simple conditions are not sufficient. The nonabelian group G of order 14 has a faithful irreducible representation $g \to R(g)$ of degree $(= R(e))$ three such that trace $R(g)$ is an integer greater than or equal to -1 for all $g \in G$. If it were possible to satisfy (3) for this group and representation then G would have a faithful representation as a doubly transitive group. But this would contradict corollary 1, since there is no integer m such that $m(m-1)$ divides 14.

5. Other Results

Hardly an issue of *Mathematical Reviews* goes by without some mention of double transitivity. Transitivity properties are among the most interesting from the fruitful and durable field of permutation groups. Interesting results beyond those which have been given here abound; but the methods are not elementary. Since he has raised the question of solvability, however, the author cannot resist including one last remark without proof, one which is perhaps illustrative of current work. Burnside [1, p. 341] has "shewn" that a *simply transitive* (transitive but not doubly transitive) group of prime degree p is of order pq where q is a prime factor of $p-1$. Since groups of order pq are solvable, it follows [9, p. 29] that *every nonsolvable transitive group of prime degree is doubly transitive.*

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Covariances of Two Sample Rank Sum Statistics

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This note presents an elementary derivation of the covariances of the $c(c-1)/2$ two-sample rank sum statistics computed among all pairs of samples from c populations.

Key words: c Sample problem; covariances, Mann-Whitney-Wilcoxon statistics; rank sum statistics; statistics.

Mann-Whitney or Wilcoxon rank sum statistics, computed for some or all of the $c(c-1)/2$ pairs of samples from c populations, have been used in testing the null hypothesis of homogeneity of distribution against a variety of alternatives [1, 3, 4, 5].[1] This note presents an elementary derivation of the covariances of such statistics under the null hypothesis.

The usual approach to such an analysis is the rank sum viewpoint of the Wilcoxon form of the statistic. Using this approach, Steel [3] presents a lengthy derivation of the covariances. In this note it is shown that thinking in terms of the Mann-Whitney form of the statistic leads to an elementary derivation. For comparison and completeness the rank sum derivation of Kruskal and Wallis [2] is repeated in obtaining the means and variances.

Let x_i^r, $r=1,2,\ldots,n_i$, $i=1,2,\ldots,c$, be the rth item in the sample of size n_i from the ith of c populations. Let M_{ij} be the Mann-Whitney statistic between the ith and jth samples defined by

$$M_{ij}=\sum_{s=1}^{n_i}\sum_{r=1}^{n_j} z_{ij}^{rs} \tag{1}$$

where

$$z_{ij}^{rs}=\left\{\begin{array}{l} 1, x_j^s > x_i^r \\ 0, x_j^s \leq x_i^r \end{array}\right\}$$

Thus M_{ij} is the number of times items in the jth sample exceed items in the ith sample. Let W_{ij} be the Wilcoxon rank sum statistic defined by

$$W_{ij}=\sum_{s=1}^{n_j} R_{ij}(x_j^s), \tag{2}$$

where $R_{ij}(x_j^s)$ is the rank of x_j^s in the combined ith and jth samples. Then M_{ij} and W_{ij} are related by

$$M_{ij}=W_{ij}-n_j(n_j+1)/2. \tag{3}$$

AMS Subject Classification: 6231.

[1] Figures in brackets indicate the literature references at the end of this paper.

Obviously $E(M_{ij}) = E(W_{ij}) - n_j(n_j+1)/2$, the variances of M_{ij} and W_{ij} are equal and the covariances among the M_{ij} are equal to the covariances among the W_{ij}.

THEOREM: *Under the null hypothesis, the Mann-Whitney form of the statistic has the following moments:*

$$E(M_{ij}) = n_i n_j/2$$

$$V(M_{ij}) = n_i n_j(n_i + n_j + 1)/12$$

$$\left.\begin{aligned} C(M_{ij}, M_{ik}) = C(M_{ji}, M_{ki}) &= n_i n_j n_k/12 \\ C(M_{ij}, M_{ki}) &= -n_i n_j n_k/12 \end{aligned}\right\} \text{ ijk } \textit{all different}$$

$$C(M_{ij}, M_{kl}) = 0 \qquad \text{ijkl } \textit{all different.}$$

PROOF: Consider the population consisting of one of each of the integers $1,2,\ldots,N_{ij} = n_i + n_j$. The population mean and variance are $(N_{ij}+1)/2$ and $(N_{ij}^2 - 1)/12$, respectively. Now let R_{ij} be the mean of a random sample of size n_j drawn without replacement from the population.

Then

$$E(R_{ij}) = (N_{ij}+1)/2, \tag{4}$$

and

$$V(R_{ij}) = \frac{(N_{ij}^2-1)}{12n_j}\left[\frac{N_{ij}-n_j}{N_{ij}-1}\right] = \frac{(N_{ij}+1)(N_{ij}-n_j)}{12n_j} \tag{5}$$

where the quantity in brackets is sometimes called the finite sample correction factor. Under the null hypothesis, the distribution of W_{ij} is identical to that of $n_j R_{ij}$. The mean and variance of M_{ij} follow immediately.

To obtain the covariances, let $M_{i(j+k)}$ be the Mann-Whitney statistic for the combined jth and kth samples compared to the ith sample. Recalling that M_{ab} is the number of times items in the bth sample exceed items in the ath sample, it is easily seen that $M_{i(j+k)} = M_{ij} + M_{ik}$. Now, since $n_{j+k} = n_j + n_k$,

$$V(M_{i(j+k)}) = V(M_{ij} + M_{ik}) = n_i(n_j + n_k)(n_i + n_j + n_k + 1)/12. \tag{6}$$

But

$$V(M_{ij} + M_{ik}) = V(M_{ij}) + V(M_{ik}) + 2C(M_{ij}, M_{ik}). \tag{7}$$

It follows that

$$C(M_{ij}, M_{ik}) = n_i n_j n_k/12. \tag{8}$$

The remaining covariances follow from the identity $M_{ij} = n_i n_j - M_{ji}$.

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(Paper 76B1&2—36

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Efficient Techniques for Unbiasing a Bernoulli Generator

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Consider the problem of operating on a sequence of i.i.d. Bernoulli variables with unknown mean p to produce a sequence of symmetric Bernoulli variables. Define the efficiency of any proposed method to be the average number of binary output digits per input digit. The following results are proved: (A) No method exists having efficiency greater than $-p\log_2 p - q\log_2 q$, where $q = 1 - p$. (B) Methods do exist with efficiency arbitrarily close to the bound just given. Examples are given, and compared with other methods in the literature. A technique for finding the methods of (B) above is given.

Key words: Bernoulli; binomial; coin—tossing; computer; efficiency; generator; random numbers; statistics.

1. Introduction

The problem of deriving unbiased Bernoulli variables from a sequence of independent and identically distributed Bernoulli variables with mean p has received attention repeatedly in the literature. The most recent appearance known to this author is a paper by Simons and Hoeffding [4],[1] wherein are given references to other work. The approach adopted by Simons and Hoeffding concentrates on the sample sequences, and investigates stopping rules and decision rules which specify when a sample sequence is terminated with the issuance of a (binary) output digit, and which digit is output. One rule is "better" than another, in their terminology, if it stops as soon as the other for any sample sequence, and sooner for at least one sequence. They define the class of "even" procedures, and find the best even procedure, which they denote by Q_2. They then find a better (noneven) procedure Q_3, a lower bound for the expected length of the sample sequence, a proof that there is no procedure which is as good as or better than all others, and a procedure Q_4 which is better than Q_3 for small p values. Incidentally, they start by introducing the procedure of von Neumann [5], denoted Q_1, which consists of taking successive pairs of digits until a mixed pair (i.e., a pair containing both a 0 and a 1) is obtained, and outputting the second digit of the pair.

In this paper, the efficiency of a procedure is defined in a long-run sense, as the average number of output digits per input digit. Thus a procedure which does not terminate as quickly as one of the Q_i above, for certain sample sequences, and yet produces output digits at a higher average rate, is considered more efficient. For reference, note that the efficiency of Q_1 is $pq \leqq 1/4$ (where $q = 1 - p$).

The efficiency of Q_2 is shown by Simons and Hoeffding to be

$$(1/2)\prod_{i=1}^{\infty}(1+p^{2^i}+q^{2^i})^{-1},$$

which is less than

$$(1/2)(1+p^2+q^2)^{-1}(1+p^4+q^4)^{-1} \leqq 8/27 < 1/3.$$

AMS Subject Classification: Primary 65C10, Secondary 60-04.

* This paper is based on work largely completed while the author was with Research Analysis Corporation, McLean, Va. 22101.

[1] Figures in brackets indicate the literature references at the end of this paper.

As a matter of fact, the efficiency of Q_3 is also less than 1/3. The lower bound for the expected size of the sample sequence for any procedure which looks at sample sequences in the way considered by these authors—i.e., which consists of a stopping rule and a rule for specifying an output digit whenever the sequence stops—is given by the authors as $p^{-1}q^{-1} - 1$. This immediately leads to an upper bound for the efficiency of such procedures, namely $pq(1-pq)^{-1}$. This function in turn takes on its maximum value at $p = 1/2$, for which it is 1/3.

The procedure Q_2 defined by Simons and Hoeffding is equivalent to one presented by this author [2] at the 1964 Institute of Mathematical Statistics Meeting at Amherst, namely, stop as soon as a block from $(k-1)2^n+1$ to $k \cdot 2^n$, for any positive integers k and n, is evenly split between zeros and ones, and output the last digit; then start again.

The procedure Q_3 is an ingenious improvement over Q_2. Simons and Hoeffding noted that there are circumstances when it is certain that the next digit (whatever it turns out to be) will result in termination—and so there is no need to take that last digit, if one simply decides in the beginning to output the next-to-last digit (whether or not the last digit is needed). An example is in order at this point. The sequences 00001100 and 00000011 are equally likely, as are the sequences 00000010 and 00000001. Under Q_2, these sequences will lead to termination, with output 0,1,0,1, respectively. If we were to output the next-to-last digit at termination, the output would be 0,1,1,0, respectively. Now note that in all but the last sequence, we know before the eighth digit is taken that it will be the last digit. The procedure Q_3 merely takes advantage of this fact, by stopping as soon as it becomes known that termination will occur on the next digit. (Obviously, the *last digit taken* is the output of Q_3 whenever Q_3 stops sooner than Q_2, while the next-to-last digit is the output when Q_3 stops at the same point as Q_2.)

In the next section, an intuitive derivation is given of a different kind of procedure, which grew out of a suggestion by Herman Rubin to the author during the Amherst meeting previously mentioned. Following sections present a rigorous evaluation of the method, examples, and proofs.[2]

2. A Different Approach

Consider again the von Neumann procedure above. It consists of taking pairs of observations, ignoring any pair consisting of two zeros or of two ones, and putting out one digit if a mixed pair occurs. Now two zeros or two ones can happen in only one way each, but a mixed pair can be either 01 or 10—and one is as likely as the other. Thus another way of looking at the von Neumann procedure is in terms of rearrangements of the result that occurred: since there is only one arrangement possible for the sequence 11, no output can be obtained, but a mixed sequence can happen in either of two equally likely ways which are assigned the (output) values 0 and 1 (in arbitrary order).

Now let us extend this technique. Suppose four input digits are examined together. If all are alike, then no rearrangements are possible—hence no output. If there is exactly one 0, or three 0's, then there are four equally likely rearrangements—1110, 1101, 1011, 0111 in the former case, and the complements in the latter. We can assign output sequences 00, 01, 10, and 11 to these four possibilities, and thus obtain two output digits. But what if there are two 1's and two 0's? Then there are $\binom{4}{2} =$ six equally likely possibilities. We can take four of them, and derive two binary digits as above; but we can derive only one binary digit when one of the remaining two possibilities occurs. Thus we will get 0, 1, or 2 output digits for 4 input digits, with probabilities (p^4+q^4), $2p^2q^2$, and $4p^3q + 4pq^3 + 4p^2q^2$ respectively, for an average of $2pq - 3p^2q^2/2$ output digits per input digit. This value is at least $(13/8)pq$, incidentally, so that this procedure is considerably more efficient than the von Neumann procedure, even for groups of four; it can be made still more efficient for larger groups, as will be seen.

There is something basically unsatisfying about this procedure as just outlined. Consider again the simple example above with group size 4. There are six arrangements of two zeros and two ones,

[2] The author has recently learned of a related paper by Prof. Peter Elias, to be published in the *Annals of Mathematical Statistics*.

which is more than the four arrangements of one zero and three ones; yet the number of output digits is (on the average) only 5/3, which is less than the two digits obtained for the case of one zero and three ones. The reason is obvious: 6 is not a power of 2. (For 7 arrangements, the expected number of digits is less: only 10/7.) One possible way to improve things is to save the set with six arrangements, and combine it with the next set having six arrangements. We thus have 36 possibilities (for the combined set): we can get 5 ($=\log_2 32$) digits with probability 32/36, and 2 digits with probability 4/36, for an average of 7/3 digits per set. We could go further, and find a higher power of 6 which is closer to a power of 2; we could then get even more digits per set.

This simple example embodies the key idea of the method. In the next section, a more complete discussion of the technique is presented. Two lemmas and a theoretical upper bound for efficiency are given, and used to show the existence of procedures with efficiencies arbitrarily close to the upper bound. This bound is then compared to the efficiencies of the Q^i and of selected examples of the procedures, in the following section.

3. Permutation Methods

Consider sets of n independent and identically distributed Bernoulli variables with mean p. Any such set will have from zero to n 1's. The probability that a set will have x 1's is $\binom{n}{x}p^x q^{n-x}$ and a set with x 1's can have any of $\binom{n}{x}$ equally likely configurations. Let $m=\binom{n}{x}$. If m is a power of 2, one can number all of the m configurations with $(\log_2 m)$—digit binary numbers, in such a way that each such number is used exactly once. Then when one of the m configurations occurs, one can simply output the $(\log_2 m)$ digits in the number assigned to that configuration. If m is not a power of 2, a record may be kept of which configuration occurred (i.e., a number between 1 and m may be recorded). In future sets of n digits, every value of m will (with probability one) come up repeatedly, so that one obtains sequences of numbers between 1 and m (inclusive)—one sequence for each distinct value of m. In each sequence, each number is equally likely to take on any of the m values, of course. With $j(m)$ numbers, each equally likely to have any value between 1 and m, one has $m^{j(m)}$ equally likely possibilities. It will be shown below that the integer $j(m)$ can be chosen so that $m^{j(m)}$ is only slightly larger than a power of 2. Thus with high probability one obtains very nearly $\log_2 m$ digits for each of the $j(m)$ sets.

A "permutation procedure" is a procedure which operates as just outlined. If n is the size of the set upon which the procedure is based, one needs to specify values $j(m)$ for each $m=\binom{n}{x}$, for $x=0,1,2,\ldots,n$. (Note that $\binom{n}{x}=\binom{n}{n-x}$.) The j in Lemma 1 and Theorem 1 below is the $j(m)$ just considered, and the i below is the integer such that 2^i is almost as big as $m^{j(m)}$; i.e., $i=[j(m)\log_2 m]$. Let Eu denote the mathematical expectation of the random variable u. Then we have:

THEOREM 1: *Given $\epsilon>0$, a sequence of independent and identically distributed Bernoulli variables, and an integer* $n>1$, *There exists a permutation procedure based on sets of* n, *with efficiency at least* $n^{-1}\,\mathrm{Elog}_2\binom{n}{x}-\epsilon$.

To prove this result, we will make use of the following lemma, proved in the Appendix:

LEMMA 1: *Given a real number* k, *and an* $\epsilon>0$, *there exist integers* i *and* j *such that* $0\leqq k-(i/j)<(\epsilon/j)$.

PROOF OF THEOREM 1: For each possible value of $m=\binom{n}{x}$, Lemma 1 implies the existence of integers i and j such that $0\leqq \log_2 m-(i/j)<\epsilon/2j$. A permutation procedure using these integers will have efficiency given by

$$E_n=n^{-1}\sum_{x=1}^{n-1}D_x\binom{n}{x}p^x q^{n-x},$$

where D_x denotes the expected number of output digits from a set of n digits having x 1's. Now D_x is at least (i/j) times the probability of getting i digits from a group of j such sets, where $i=[\log_2 m^j]$. This probability is $2^i/m^j$. From the inequality above,

$$0\geqq i-j(\log_2 m)>-\epsilon/2,$$

or equivalently,

$$1 \geqq 2^i/m^j > 2^{-\epsilon/2} > e^{-\epsilon/2} > 1-(\epsilon/2);$$

and also, $(i/j) > \log_2 m - \epsilon/2$. Therefore

$$\begin{aligned} E_n &> n^{-1} \sum_{x=0}^{n} (1-\epsilon/2)\,(\log_2 \tbinom{n}{x} - \epsilon/2)\,\tbinom{n}{x} p^x q^{n-x} \\ &= n^{-1} E \log_2 \tbinom{n}{x} - (\epsilon/2n)\{E \log_2 \tbinom{n}{x} + 1\} + \epsilon^2/4n \\ &> n^{-1} E \log_2 \tbinom{n}{x} - \epsilon. \end{aligned}$$

(The last step is valid because $\binom{n}{x}2^{-n}$, being a probability, is at most unity, implying that $\log_2\binom{n}{x} \leqq n$.) Q.E.D.

Lemma 2, also proved in the Appendix, asserts that

$$\lim_{n\to\infty} n^{-1} E \log_2 \tbinom{n}{x} = -p\log_2 p - q\log_2 q.$$

This then is the limiting efficiency of permutation procedures. One can do no better, as evidenced by the following.

THEOREM 2: *Any procedure for generating symmetric Bernoulli variables from a stream of independent and identically distributed Bernoulli variables has efficiency at most* $-(p\log_2 p + q\log_2 q)$, where p is the mean value of the independent and identically distributed variables.

PROOF: Kullback [1, p. 13] proves that for a random variable X, and a statistic Y (a function of X), the average information in an outcome X is at least as great as that in the derived statistic Y. The "information" referred to in this theorem is the information for discrimination in favor of one hypothesis, H_1, against another one, H_2. Let X be an arbitrarily long (finite) sequence of independent and identically distributed Bernoulli variables, and Y the derived sequence of symmetric Bernoulli variables. Let n be the length of the X-sequence, and M (a random variable) the length of the Y-sequence; let $m = E(M)$. Let H_1 be the hypothesis that the mean of the elements of X is p_0; in terms of Y, this hypothesis implies that the mean of the elements of Y is 1/2, regardless of p_0. Let H_2 be the hypothesis that the mean of the X elements is in the closed unit interval. Then (in Kullback's notation)

$$\begin{aligned} I(1{:}2;\, X) &= -\sum_{s=0}^{n} \tbinom{n}{s} p^s q^{n-s} (s \log_2 p_0 + (n-s) \log_2 q_0) \\ &= -n(p \log_2 p_0 + q \log_2 q_0) \\ &\leqq -n(p \log_2 p + q \log_2 q). \end{aligned}$$

Similarly,

$$I(1{:}2;\, Y) = E\{-\sum_{t=0}^{M} \tbinom{M}{t} 2^{-M}\{-t-(M-t)\}\} = E(M) = m.$$

By Kullback's theorem, $m \leqq -n(p\log_2 p + q\log_2 q)$. But the efficiency of the procedure under consideration is simply the limit of the ratio m/n as $n \to \infty$; thus the efficiency is limited by $-(p\log_2 p + q\log_2 q)$. Q.E.D.

Finally, we note that there is a simpler approach to the problem of attaining high efficiency. As can be seen from table 2, the limit of attainable efficiency for small n is not very high. Since one approaches this limit by taking larger values of j, using more storage space and a more complicated rule for determining the output, it would seem worthwhile to consider direct methods using a larger value of n. In other words, if it is necessary to save (an average of) ten sets of $n = 8$ digits to come close to the limit $E\log_2\binom{8}{x}$ derived above, we ought to compare with one set of $n = 80$. How does one use such a set? Suppose x is the number of 1's that occur. Then $m = \binom{80}{x}$. One numbers the m configurations arbitrarily from 0 to $m-1$, and sets off a group of size equal to the largest power of

2 less than (or equal to) m; then from what is left, a group of size equal to the largest power of 2 less than or equal to the number remaining; then with what is still left, repeat the procedure, until the entire group of m is exhausted. If the outcome should fall in a group of size 2^i, one gets i digits. It is shown at the end of the Appendix that this procedure will attain an efficiency of at least $n^{-1}(E(\log_2(\binom{n}{x}))-2)$, which approaches the value $n^{-1}E(\log_2(\binom{n}{x}))$ obtained above as n gets large.

A few words are in order, at this point, about the practical implications of the techniques described above. Specifically, a method of obtaining the proper binary output stream from the input stream is needed. There are two steps to be accomplished: first, derive an index number for the input sequence, which identifies which of the equally likely configurations has occurred; and second, derive the appropriate output sequence from this index number. They will be considered in turn.

Consider first the simple case of a set of n input digits, and let x be the number of 1's among the n digits. As before, let $m=\binom{n}{x}$. We will derive an index number between 0 and $m-1$ inclusive which gives the position of the actual configuration in a sequence of the m possible configurations, arranged in order of increasing magnitude (considered as binary numbers). Let z denote the actual sequence (of x 1's and $(n-x)$ 0's). Check the first bit. If it is 1, then z is not one of the $\binom{n-1}{x}$ combinations starting with 0. Since these are precisely the first $\binom{n-1}{x}$ combinations, add the binomial coefficient $\binom{n-1}{x}$ into a counter (which will eventually contain the index number corresponding to z). If the first bit is 0, then z is one of the first $\binom{n-1}{x}$ combinations, so don't add anything into the counter. Now check the next bit. If 1, add in the appropriate number – either $\binom{n-2}{x}$ or $\binom{n-2}{x-1}$, depending

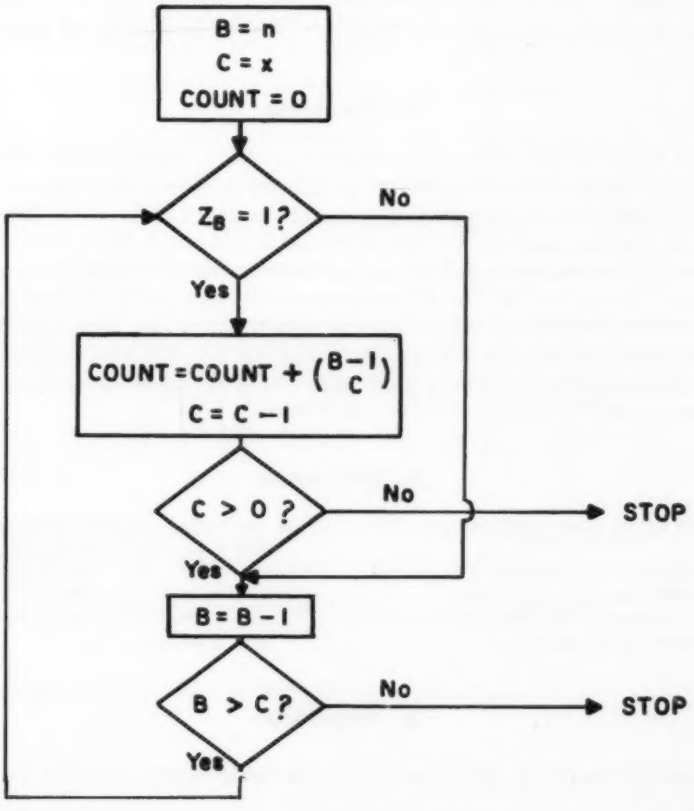

FIGURE 1. *Flow chart for procedure assigning an index number between* 0 *and* m − 1 *inclusive to a sequence* $Z_nZ_{n-1}Z_{n-2}\ldots Z_1$ *containing* x *ones and* n − x *zeros;* $m=\binom{n}{x}$.

on whether the first bit was a 0 or a 1. Continue in this fashion through all the bits of z, stopping as soon as the remaining bits of z are known to be all alike. This is really a fairly simple procedure to program: a flow chart is shown in figure 1.

For a permutation procedure, one has several (say j) sets of n digits, each set being one of m configurations. One assigns a number between 0 and $m-1$, as above, to each set of n. Then one assigns the number $n_1 + n_2 \cdot m + n_3 \cdot m^2 + \ldots + n_j \cdot m^{j-1}$ to the combination of j sets of n, where n_i is the number assigned to the ith set of n. This is of course a number between 0 and $m^j - 1$, as required.

Now consider the determination of the output sequence, given the index number of the input. Let y denote the index number of the actual result, and M the number of equally likely results (of which this result is one). Define i by $2^i \leqq M < 2^{i+1}$. Then there are $i+1$ bits in the binary representation of M:

$$M = \sum_{j=0}^{i} a_j 2^j,\ a_i = 1;\ a_j = 0 \text{ or } 1, \text{ all } j < i$$

If $y < 2^i$, we should obtain i bits; if $2^i \leqq y < 2^i + 2^{i_1}$ where i_1 is the largest $j (< i)$ with $a_j = 1$, we should obtain i_1 bits; etc. Now $y < 2^i$ precisely when the ith bit of y is 0 (considering the least significant bit as the $0th$ bit). And conveniently, if the ith bit of y is 0, then the i less significant bits are symmetric Bernoulli variables—because the possible values of these bits, when y has the ith bit equal to zero, consist of all the binary numbers from 0 to $2^i - 1$. Similarly, $2^i \leqq y < 2^i + 2^{i_1}$ precisely when the ith bit of y is 1 and the i_1st bit is 0; and when this is true, the lower-order i_1 bits are symmetric Bernoulli variables. This process can be continued through all the bits of y. In summary, the procedure is to check in turn all bits of y corresponding to non-zero bits of M, beginning with the most significant; as soon as one of the checked bits is found to be 0, stop, and output the remaining bits of y.

4. Examples

The primary results of this section are contained in table 1, which lists the efficiency of each of several techniques. Among those evaluated are Q_1, Q_2, and Q_3, from Simons and Hoeffding; the simple procedures outlined above (permutation procedures with $j=1$) for $n=4$, 8, 20, 50, and 1024, denoted by S_4, S_8, S_{20}, S_{50}, and S_{1024} and one permutation procedure denoted by P_1. Of course, P_1 needs to be specified: we require n, and the appropriate number of j values. These are given in table 2. Note that for $x=3$ or 5, $m=56$. The 56 sequences can be (arbitrarily) divided into two groups of 28, and index numbers from 0 to 27 can be assigned within each group. Then when $x=3$ or 5 occurs, one binary digit can be output, 0 or 1 according to which group of 28 contains the result; the index number can be treated just as if it were a result with $x=2$ or 6. This of course decreases the amount of storage necessary and the average time between outputs.

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6. Appendix

Contained herein are proofs of the two lemmas, and of the lower bound for the efficiency of the $S-$procedures.

LEMMA 1: *Given a positive number* k, *and* $\epsilon > 0$, *there exist positive integers* i *and* j *such that* $0 \leqq k - (i/j) < \epsilon/j$.

PROOF: If k is rational, there exist integers i, j such that

$$k - (i/j) = 0.$$

Otherwise, by Theorem 4.2, page 44 of [3], there exist positive integers i, j such that $|jk - i| < \epsilon$. If $jk - i > 0$, we are finished. Otherwise $-\epsilon < jk - i < 0$; let

$$m = [(i - jk)^{-1}] \leqq \frac{1}{i - jk}.$$

Then

$$m(jk - i) \geqq -1 > (m+1)(jk - i) > m(jk - i) - \epsilon,$$

so that

$$0 \leqq (mj)k - (mi - 1) < \epsilon. \qquad \text{Q.E.D.}$$

LEMMA 2: *Let* x *be the number of successes in* n *independent and identically distributed Bernoulli trials, with probability* p *of success. Then*

$$\lim_{n \to \infty} \{n^{-1} E \log_2 \binom{n}{x}\} = -p \log_2 p - q \log_2 q.$$

PROOF: (All logs in the sequel are to the base 2.) By Stirling's approximation,

$$\log \binom{n}{x} = n \log n - x \log x - (n - x) \log(n - x) - (1/2) \log(x - x^2/n) + O(n^{-1/2}),$$

for $1 \leqq x \leqq n - 1$. Dividing by n, and substituting $np + y$ for x, we have

$$\begin{aligned} n^{-1} \log \binom{n}{x} &= \log n - (p + y/n) \log(np + y) - (q - y/n) \log(nq - y) + O(n^{-1/2}) \\ &= -\log\{(p + y/n)^{p + y/n} (q - y/n)^{q - y/n}\} + O(n^{-1/2}) \\ &= -p \log p - q \log q - (y/n) \log(p/q) \\ &\quad - (p + y/n) \log(1 + y/np) - (q - y/n) \log(1 - y/nq) + O(n^{-1/2}). \end{aligned}$$

Now y is a random variable, with mean zero and variance npq, so that y/n has mean zero and variance pq/n. Since $\log \binom{n}{0} = \log \binom{n}{n} = 0$, we can write

$$\begin{aligned} n^{-1} E \log \binom{n}{x} = \Sigma' \{ &-p \log p - q \log q - (y/n) \log(p/q) \\ &- (p + y/n) \log(1 + y/np) - (q - y/n) \log(1 - y/nq) \\ &+ O(n^{-1/2}) \} P_{np+y}, \end{aligned}$$

where the prime on the summation indicates that the terms corresponding to $x = 0$ and to $x = n$ are omitted, and where P_{np+y} is the appropriate binomial probability. Thus

$$n^{-1} E \log \binom{n}{x} = -p \log p - q \log q - E_1 - E_2 + O(n^{-1/2}),$$

where

$$E_1 = \Sigma'(p + y/n) \log(1 + y/np) P_{np+y}$$

and

$$E_2 = \Sigma'(q - y/n) \log(1 - y/np) P_{np+y}.$$

All that remains is to show that E_1 and E_2 approach 0 as n approaches ∞. The proof will be given for E_1, that for E_2 being essentially the same.

Given a small positive quantity ϵ, we want to find a value n_0 such that for any $n > n_0$, $|E_1(n)| < \epsilon$. Because $0 < p + (y/n) < 1$,

$$|E_1| < \Sigma' |\log(1 + y/np)| P_{np+y}.$$

Now y has mean zero and variance npq. Thus by Chebychev's inequality, we have for any $a > 0$ the relation

$$Pr(|y/np| > a\sqrt{q/np}) \leq a^{-2}.$$

Therefore for $a\sqrt{q/np} < 1$,

$$|E_1| < \max \{\log (1 + a\sqrt{q/np}), |\log (1 - a\sqrt{q/np})| + a^{-2} \max |\log (1 + y/np)|$$
$$= |\log (1 - a\sqrt{q/np})| + a^{-2} \max \{|\log 1/np|, |\log (n-1)/np|\}.$$

Consider only $\epsilon < 1/2$, and $n > p^{-2}q^{-1}$. Then $np > p^{-1}q^{-1} > p^{-1} > (n-1)/np > p^{-1}(1 - p^2q) = p^{-1} - pq > 1$, so $|\log 1/np| = \log np > |\log (n-1)/np|$. Take $a = (\epsilon/4)(np/q)^{1/2}$. Then

$$|\log (1 - a\sqrt{q/np})| = |\log (1 - \epsilon/4)| < \epsilon/2,$$

and $a^{-2} \log np < 16 (\log np)/np\epsilon^2$. Now pick n_0 large enough that $(\log n_0 p)/n_0 p < \epsilon^3/32$. Then for $n > n_0$, $a^{-2} \log np < \epsilon/2$, so $|E_1| < \epsilon$.

Q.E.D.

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Checks of the Algebraic Matrices for the Configurations $(d+s)^n p$*

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By constructing suitable combinations of algebraic matrices enabling the resulting eigenvalues to be predicted theoretically, checking parameters were obtained for 32 interaction parameters of the configuration $(d+s)^n p$.

Key words: Checking parameters; $(d+s)^n p$; electrostatic interaction; relative phases; spin-orbit interaction.

1. Introduction

The algebraic matrices of $(d+s)^n p$ comprise the electrostatic and spin-orbit interaction matrices of the configurations $d^n p$, $d^{n-1} sp$ and $d^{n-2} s^2 p$, the matrices of the correction parameters representing two and three body interactions of the core d electrons, as well as the interactions between configurations. The energy matrix (for a particular n) is then a linear combination of these matrices, the coefficients of which are parameters usually obtained empirically by fitting the experimental levels to the eigenvalues of the energy matrix. In a previous paper, [1][1], the role and significance of the various parameters was discussed.

The algebraic matrices of $(d+s)^n p$ were constructed and checked by the author with the purpose of using them to explain and predict the spectra of the configurations $(3d+4s)^n 4p$ in neutral and singly-ionized atoms of the iron group. The problem of constructing the matrices was considered previously [2-4]. By using the eigenvalues of the Casimir operator for the group SU_3, checking parameters were obtained for 12 electrostatic interaction matrices of the configuration $(d+s)^n p$, [1]. The purpose of the present paper is to obtain checking parameters for the remaining interaction parameters. The checked matrices of the configurations $(d+s)^n p$, for all permissible n, are available and can be obtained by request.

All checks are based upon the construction of such algebraic matrices so that the eigenvalues of the resulting matrix may be predicted theoretically. If we know the values of the parameters required in order to form the desired combinations, then we insert these values as coefficients multiplying the algebraic matrices found on tape, and diagonalize the resulting matrix. If the obtained eigenvalues differ from those predicted theoretically then it is usually not too difficult to find the reason for the discrepancy because in a particular check only a few of the parameters are utilized.

2. The Check ^1P

In this check we do not obtain exact numerical values of the predicted eigenvalues, only the number of eigenvalues differing from zero in any particular term of $(d+s)^n p$. We first choose the

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[1] Figures in brackets indicate the literature references at the end of this paper.

parameters of the configurations $(d+s)p$ in such a way that as many terms as possible of $(d+s)p$ should have vanishing eigenvalues. By adding two parameters to the electrostatic parameters of $(d+s)p$, it is found that we can cause all the eigenvalues of $(d+s)p$, except one, to equal zero. Since the configuration $(d+s)^n p$ can be obtained from $(d+s)p+(d+s)^{n-1}$, it is most convenient to choose the nonvanishing term of $(d+s)p$ with lowest L and S, i.e., ^{1}P. If, furthermore, the rank of the 2×2 matrix of ^{1}P in $(d+s)p$ is equal to 1, then when considering a particular term of $(d+s)^n p$, the rank of its matrix equals the number of terms of $(d+s)^{n-1}$, which in combination with ^{1}P yield the desired term of $(d+s)^n p$. By choosing ^{1}P of $(d+s)p$ to be nonvanishing the matrices of $(d+s)^n p$ are of the lowest rank and thus have a maximal number of vanishing eigenvalues.

In the configuration dp there are the six terms ^{1,3}PDF and in the configuration sp the two terms are ^{1,3}P. We may be inclined at first to search for parameters which would cause all the terms of $(d+s)p$ to vanish identically. However, for this purpose we need 11 parameters to solve the resulting 10 homogeneous equations, whereas the number of electrostatic parameters of $(d+s)p$ is only 8, i.e., $F_0, F_2, G_1, G_3, F_0', G'_{ps}, J$ and K. It is, however, possible to add two additional parameters which are independent of the previous six. We thus add to each term of $(d+s)p$

$$\epsilon\,(l_1 \cdot l_2)+\mu(s_1 \cdot s_2).$$

For d-p this contribution equals

$$\begin{aligned}&\epsilon/2\,[L(L+1)-l_d(l_d+1)-l_p(l_p+1)]\\&+\mu/2\,[S(S+1)-s_d(s_d+1)-s_p(s_p+1)]\\&=\epsilon/2\,[L(L+1)-8]+\mu/2\,[S(S+1)-3/2]\,.\end{aligned}\tag{1}$$

For $s-p$ we have

$$\begin{aligned}&\epsilon/2\,[L(L+1)-l_s(l_s+1)-l_p(l_p+1)]+\mu/2\,[S(S+1)-s_s(s_s+1)-s_p(s_p+1)]\\&=\mu/2\,[S(S+1)-3/2].\end{aligned}\tag{2}$$

Here L and S are the net angular momentum and spin respectively, of a particular term of $(d+s)p$. We note that here the definitions of ϵ and μ differ slightly from those of the parameters λ and δ introduced in a previous paper, [1].

From p. 200 TAS [5], and (1) the terms of dp can be written as

$$\begin{aligned}^1\mathrm{P}&=F_0+7F_2+G_1+21G_3-3\epsilon-3/4\,\mu\\^3\mathrm{P}&=F_0+7F_2-G_1-21G_3-3\epsilon-1/4\,\mu\\^1\mathrm{D}&=F_0-7F_2-3G_1+7G_3-\epsilon-3/4\,\mu\\^3\mathrm{D}&=F_0-7F_2+3G_1-7G_3-\epsilon+1/4\,\mu\\^1\mathrm{F}&=F_0+2F_2+6G_1+G_3+2\epsilon-3/4\,\mu\\^3\mathrm{F}&=F_0+2F_2-6G_1-G_3+2\epsilon+1/4\,\mu\,.\end{aligned}\tag{3}$$

The two terms of sp are

$$\begin{aligned}^1\mathrm{P}'&=F_0'+G'_{ps}-3/4\,\mu\\^3\mathrm{P}'&=F_0'-G'_{ps}+1/4\,\mu\,.\end{aligned}\tag{4}$$

In (3) we have six equations with six independent parameters. Thus, we can let five of the equations to equal zero and the sixth to differ from zero. As stated previously, the term chosen to differ from zero is ^{1}P. On solving (3) we obtain parameters proportional to the following:

$$F_0=35, F_2=5, G_1=4, G_3=6, \epsilon=-15 \text{ and } \mu=60\,.\tag{5}$$

The value of ^1P is then equal to 200.

For sp we require that $^3P'$ be equal to zero, and thus form (4) and (5)

$$\begin{aligned} {}^1P' &= F_0' + G'_{ps} + 45 \\ 0 &= F_0' - G'_{ps} + 15 . \end{aligned} \tag{6}$$

The interaction between configurations dp and sp is given by equation (7), [4]. For ^1P this equals $-\sqrt{2}(J+K)$, whereas for ^3P the interaction equals $-\sqrt{2}(J-K)$. Furthermore, since the matrix of ^3P is to be identically equal to zero, the nondiagonal contribution also vanishes.

Thus

$$J = K. \tag{7}$$

The matrix of $(dp+sp)$ then reduces to the 2×2 matrix of ^1P which is given by

$$^1P = \begin{pmatrix} 200 & -2\sqrt{2}J \\ -2\sqrt{2}J & 2F_0' - 30 \end{pmatrix}. \tag{8}$$

For the above matrix to be of rank 1 its determinant must vanish. Then,

$$50\ F_0' - 750 = J^2. \tag{9}$$

We can choose

$$J = K = 10\sqrt{10}. \tag{10}$$

Then

$$F_0' = 35. \tag{11}$$

By (6)

$$G'_{ps} = 50. \tag{12}$$

Then by (5)

$$S' = F_0' - F_0 = 0. \tag{13}$$

With the choice of parameters given by (5), (10), (11), (12), and (13) the only nonvanishing eigenvalue of $(d+s)p$ is that of ^1P. By inserting the values of F_0' and J into (8) the numerical value of this eigenvalue is found to equal 240.

Instead of the matrices ϵ and μ we must use the matrices of α, λ, and δ since the latter are on tape for all $(d+s)^n p$. From the definitions of the parameters α, λ, and δ and noting that the core of $(d+)p$ is $(d+s)$, we have from p. 32 [1]

$$\begin{aligned} \lambda &= \epsilon/2 = -7.5 \\ \alpha &= -\epsilon/2 = 7.5 \\ \delta &= \mu/2 = 30. \end{aligned} \tag{14}$$

In addition there is a constant contribution $-\epsilon/2\, l_p(l_p+1) - \mu/2\, s_p(s_p+1) = -7.5$.

Thus, we must add -7.5 to the height of $(d+s)^n p$. Now, for dp

$$A = F_0(dp) = 35. \tag{15}$$

For $d^n p$ there are only n interactions $d-p$ as the parameters B and C of the interaction $d-d$ are set equal to zero in this check. Furthermore, taking into account the above contribution of -7.5 we have

$$A(d^n p) = 35n - 7.5. \tag{16}$$

Since in addition to the $d-d$ interaction, the $d-s$ interaction is also equal to zero, and $F_0'(ps)$ equals $F_0(dp)$, we have that the A's of the configurations d^np and $d^{n-1}sp$ are equal. Thus $S'(d^{n-1}sp)$ equals zero.

For $d^{n-2}s^2p$ we must take into account the constant $-G'_{ps}$ for each term, which is not put on tape.

Then we have

$$S''=-G'_{ps}=-50. \tag{17}$$

Thus, finally we have the following parameters for the check ^{1}P/of the configuration $(d+s)^np$:

$$\begin{aligned}
&A(d^np)=35n-7.5\\
&S' \quad =0\\
&S'' \quad =-50\\
&F_2=F_2'=F_2''=5\\
&G_1=G_1'=G_1''=4\\
&G_3=G_3'=G_3''=6\\
&G'_{ps} \quad =50\\
&\alpha=\alpha' \ =\alpha''=7.5\\
&J=J' \ =K=K'=10\sqrt{10}\\
&\lambda=\lambda' \ =\lambda''=-7.5\\
&\delta=\delta' \ =\delta''=30.
\end{aligned} \tag{18}$$

For the configuration $(d+s)^2p$, the matrices of α', λ'' and δ'' are constants and were thus not put on tape. However, we must take these parameters into account by incorporating them into S' and S''.

Thus,

$$\begin{aligned}
S'(dsp)&=0+6\times 7.5=45\\
S''(s^2p)&=-50+3/4\times 30-2\times 7.5=-42.5.
\end{aligned} \tag{19}$$

In the configuration $(d+s)^3p$, the matrix of α'' is a constant and thus not on tape.

Therefore,

$$S''(ds^2p)=-50+6\times 7.5=-5. \tag{20}$$

It should be noted that in this check there is considerable freedom in choosing the checking parameters. Firstly, the parameters of (5) can be replaced by parameters whose values are proportional to those of (5). Also we can choose the values of F_0' and J in any convenient way as long as (9) is satisfied. Finally, it is not necessary to cause ^{1}P to be the only nonvanishing term. By the same token we can choose ^{3}P to be nonvanishing. The latter choice would only increase the rank of the matrix of a particular term of $(d+s)^np$, which is obtained by the combination of $(d+s)^{n-1}+{}^3$P.

Since we have chosen the matrix of ^{3}P to be vanishing, J is equal to K in this check and thus nondiagonal elements of the interaction between configurations which are proportional to $(J-K)$ can be wrong and we would not notice this fact from the ^{1}P check. However, the matrices of J are checked independently in the Casimir Check [1], and thus if there is an error of the nature mentioned above we would detect it from the Casimir Check.

Since the ^{1}P Check includes the parameters G_1,G_3, K and K', whose matrices differ for complementary configurations [4], we may be inclined to use this check for $n>6$. However, in its

present form the ^1P Check is useless for $n > 6$. This follows from the fact that for $n > 6$ the number of possibilities of combining $(d+s)^{n-1}$ with ^1P of $(d+s)p$ to yield a particular term of $(d+s)^n p$ is greater than the order of the matrix considered. Thus, the check ^1P for $(d+s)^n p$, where $n > 6$, must be modified by choosing $(d+s)^{11}p$ as the basic configuration rather than $(d+s)p$. Now, $(d+s)^{11}p$ can be considered as $(d+s)^{-1}p$ by the notation of section 3, [4].

From p. 229 TAS, [5], it is evident that in order for all the terms except ^1P of d^9s^2p, i.e., $d^{-1}p$, to equal zero, we must have

$$F_0(d^{-1}p) = F_2 = G_3 = \epsilon = \mu = 0. \tag{21}$$

$$G_1 \neq 0. \tag{22}$$

Then,

$$^1P = 20G_1. \tag{23}$$

For the terms of $d^{10}sp$, i.e., $s^{-1}p$, the interaction $s^{-1} - p$ is the same as the interaction $s - p$ and so we have

$$^3P' = F_0'(s^{-1}p) - G'_{ps}. \tag{24}$$

$$^1P' = F_0'(s^{-1}p) + G'_{ps}. \tag{25}$$

If, as before, we require that ^3P be equal to zero then

$$F_0'(s^{-1}p) = G'_{ps}. \tag{26}$$

$$^1P' = 2F_0'(s^{-1}p). \tag{27}$$

The interaction between the configurations $d^{10}sp$ and d^9s^2p is given by (8), [4]. For ^1P this equals $\sqrt{2}(J' - 2K')$ whereas for ^3P we have the value of $\sqrt{2}J'$ for this interaction. Since the matrix of the term ^3P must be equal to zero identically, we have

$$J' = 0. \tag{28}$$

Then the matrix of ^1P can be written as

$$^1P = \begin{pmatrix} 20G_1 & -2\sqrt{2K'} \\ -2\sqrt{2K'} & 2G'_{ps} \end{pmatrix} \tag{29}$$

We specify that the above matrix be of rank 1, and thus its determinant must vanish. This implies

$$40G_1 G'_{ps} = 8(K')^2. \tag{30}$$

As before, we can let

$$G_1 = 4, G'_{ps} = 50 \text{ and } K' = 10\sqrt{10}. \tag{31}$$

Then,

$$F_0'(s^{-1}p) = G'_{ps} = 50. \tag{32}$$

The single nonvanishing eigenvalue of the matrix (29) is then equal to 180.

For $(d+s)^{11}p$ the following parameters enter the ^1P Check:

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$$
\begin{aligned}
&A(d^9s^2p) = A(d^{-1}p) = 0\\
&A'(d^{10}sp) = A'(s^{-1}p) = 50\\
&S' \quad\; = 50\\
&G_1 \quad\; = 4\\
&G'_{ps} \;= 50\\
&K' \quad\; = 10\sqrt{10}.
\end{aligned}
\tag{33}
$$

In order to obtain the parameters for $(d+s)^{12-n}p$ where $n < 6$, we write $(d+s)^{12-n}p$ as $(d+s)^{-n}$, i.e., the configuration comprising n holes in the $(d+s)$ shells.

$$
\begin{aligned}
&A(d^{10-n}s^2p) = A(d^{-n}p) = nA(d^{-1}p) = 0\\
&A'(d^{11-n}sp) = A'(d^{-[n-1]}s^{-1}p) = A'(d^{-[n-1]}p) + A'(s^{-1}p) = 0 + 50 = 50\\
&A''(d^{12-n}p) = A''(d^{-[n-2]}s^{-2}p) = A''(d^{-[n-2]}p) + 2A'(s^{-1}p) - G'_{sp} = 0 + 100 - 50 = 50.
\end{aligned}
\tag{34}
$$

Thus,

$$
\begin{aligned}
A \;&= 0\\
S' \;&= 50\\
S'' &= 50\\
G_1 &= 4\\
G'_{ps} &= 50\\
K \;&= K' = 10\sqrt{10}.
\end{aligned}
\tag{35}
$$

In this check we have the result that for a particular term of $(d+s)^{-n}p$, the rank of the matrix equals the number of terms of $(d+s)^{-(n-1)}$, which can be combined with ^{1}P to yield the particular terms of $(d+s)^{-n}p$. This result is exactly analogous to the previous result since the configuration $(d+s)^{-n}$ has exactly the same terms as $(d+s)^n$, and similarly for $(d+s)^np$ and $(d+s)^{-n}p$. Tables 1-4 give the ranks of the matrices for each term of $(d+s)^np$ (or equivalently for each term of $(d+s)^{-n}p$).

As an example, from table 3 the rank of the matrix of $(d+s)^5p^3$D, or equivalently $(d+s)^7p^3$D, is 13. From table 6, [1], the number of expected eigenvalues of ^{3}D is 29. Thus for the ^{1}P check of $(d+s)^5p$, with parameters given by (18) or for the ^{1}P check of $(d+s)^7p$, with parameters given by (32), the matrix of ^{3}D should yield 16 vanishing eigenvalues and 13 eigenvalues differing from zero. When the rank of a particular term is not specified, it is zero and thus all the eigenvalues of that term should vanish.

TABLE 1. *Predicted number of nonzero eigenvalues for* $(d+s)^{\pm 2}p$

Term	Rank
^{2}P	2
^{2}D	1
^{2}F	1

TABLE 2. *Predicted number of nonzero eigenvalues for* $(d + s)^{\pm 3}p$

Term	Rank
^1P	4
^1D	2
^1F	3
^1G	1
^1H	1
^3S	1
^3P	2
^3D	3
^3F	2
^3G	1

TABLE 3. *Predicted number of nonzero eigenvalues for* $(d + s)^{\pm 4}p$

Term	Rank
^2S	2
^2P	7
^2D	8
^2F	8
^2G	5
^2H	3
^2I	1
^4S	2
^4P	2
^4D	4
^4F	2
^4G	2

TABLE 4. *Predicted number of nonzero eigenvalues for* $(d + s)^{\pm 5}p$

Term	Rank
^1S	1
^1P	9
^1D	8
^1F	11
^1G	7
^1H	6
^1I	2
^1K	1
^3S	5
^3P	8
^3D	13
^3F	10
^3G	9
^3H	4
^3I	2
^5S	1
^5P	2
^5D	3
^5F	2
^5G	1

TABLE 5. *Predicted number of nonzero eigenvalues of* $(d+s)^6p$

Term	Rank
^2S	4
^2P	15
^2D	18
^2F	20
^2G	15
^2H	11
^2I	5
^2K	2
^4S	4
^4P	7
^4D	11
^4F	9
^4G	7
^4H	3
^4I	1
^6P	2
^6D	1
^6F	1

3. The Check Delta

With the aid of the modified ^1P Check for $(d+s)^{12-n}p$, where $n<6$, it was possible to check the matrices of G_1,[2] K and K'. However, it is still necessary to check the matrices of G_3 for these configurations. By means of the Check Delta we obtain checks of the matrices of G_1 and G_3 for $(d+s)^{12-n}p$, where $n<6$, as well as for the matrices of F_2, G_1 and G_3 for the complementary configurations of $(d+s)^{12-n}p$, i.e., $(d+s)^np$.

If we insert for the configuration dp, the values

$$F^k=G^k=(2k+1)F^0, \tag{36}$$

which are the values corresponding to a δ-interaction [6], we get nonvanishing eigenvalues only for ^1P and ^1F. If in addition we give G^k the value

$$G^k=-(2k+1)F^0, \tag{37}$$

the resulting matrices of $dp+d^9p$ vanish identically, [7] and [8].

Now, from the definitions of the parameters F_1,G_1 and G_3 [1]

$$F_0=F^0=35 \tag{38}$$

we obtain

$$F_2=5,\ G_1=7,\ G_3=3,\ G_1=-7,\ G_3=-3. \tag{39}$$

For the configurations d^np and $d^{10-n}s^2p$, the height A is given by

$$A=nF_0=35n \tag{40}$$

as only the $d-p$ interaction is taken into account in this check.

Similarly for the configurations $d^{n-1}sp$ and $d^{11-n}sp$ as well as $d^{n-2}s^2p$ and $d^{12-n}p$ we have

[2] (G_1 and G_3 are the parameters of $(d+s)^{12-n}p$, where $n<6$, to distinguish from G_1 and G_3 of $(d+s)^np$, which are on the same tape.)

$$
\begin{aligned}
A' &= (n-1)F_0 = 35(n-1) \\
A'' &= (n-2)F_0 = 35(n-2) \\
S' &= -35 \\
S'' &= -70.
\end{aligned}
\tag{41}
$$

The other parameters are given by (38), (39) and (40).

This check is useful, simple and very convenient as all the expected eigenvalues vanish. The Check Delta may be particularly helpful in the case where by chance the other checks may not locate the error.

4. The Check C, G'_{ds} and β

The primary purpose of this check is to verify that the matrices of C and β are correct. Incidentally, the matrices of G'_{ds} and δ are also checked. In our case δ has no physical significance since it is only a checking parameter, [1]. Since only the parameters C, G'_{ds}, β and δ enter this check, it is used for $n \leq 6$.

From (88′) [9], the energy of d^n, for the case of B equal to zero, is given by

$$
W = \sum_{i<j} W_{ij} = \sum_{i<j} [A + \tfrac{1}{2} - \{2(\mathbf{s}_i \cdot \mathbf{s}_j) + q_{ij}\}C]
$$

Here $\mathbf{s}_i$ and $\mathbf{s}_j$ are spins of the electrons i and j, and $\sum_{i<j} q_{ij}$ is the seniority operator Q((50), [10]). By (91) [9],

$$
2\sum_{i<j} (\mathbf{s}_i \cdot \mathbf{s}_j) = S_1(S_1+1) - \tfrac{3}{4}\, n,
$$

where S_1 is the net spin of d^n.

Since there are $n(n-1)$ pairs in the summation we have

$$
W = \frac{n(n-1)}{2} A + \left[\frac{n(n+2)}{4} - S_1(S_1+1) + Q\right]C. \tag{42}
$$

Since $\frac{n(n-1)}{2}$ is a constant for a particular $d^n p$ under consideration, we can replace A by $\frac{n(n-1)}{2} A$ as the height of the configuration, and call the result simply A. To (42) we add βQ and $\delta[S(S+1) - S_1(S_1+1)]$, where S is the spin of $d^n p$.

Then, (42) becomes

$$
W = A + \left[\frac{n(n+2)}{4} - S_1(S_1+1) + Q\right]C + \beta Q + \delta[S(S+1) - S_1(S_1+1)]. \tag{43}
$$

If we now let

$$
C = -4,\ \beta = 4 \text{ and } \delta = 4, \tag{44}
$$

we get

$$
W = A - n(n+2) + 4S(S+1). \tag{45}
$$

If, furthermore, all the other parameters of $d^n p$ are given a value equal to zero and

$$A = (n+1)^2 \tag{46}$$

we have the very simple result

$$W(d^n p) = (2S+1)^2. \tag{47}$$

For the configurations $d^{n-1}sp$ we let S, S_2 and S_1 be the net spins of $d^{n-1}sp$, $d^{n-1}s$ and d^{n-1}, respectively. Then from (43) we obtain

$$W(d^{n-1}sp) = A' + \left[\frac{(n-1)(n+1)}{4} - S_1(S_1+1) + Q'\right]C' + \beta' Q' + \delta'[S(S+1) - S_2(S_2+1)]. \tag{48}$$

By using the parameters for C', β' and δ' as given in (44) we get

$$W(d^{n-1}sp) = A' - n^2 + 1 + 4S_1(S_1+1) - 4S_2(S_2+1) + 4S(S+1). \tag{49}$$

The term $S_1(S_1+1) - S_2(S_2+1)$ can be compensated for by the coefficient of G'_{ds} which by (32), [2] equals

$$-\tfrac{1}{2}[(n-1) + 4\mathbf{S}_1 \cdot \mathbf{s}] = -\frac{(n-1)}{2} - S_2(S_2+1) + S_1(S_1+1) + \tfrac{3}{4}. \tag{50}$$

Then, by giving G'_{ds} a value equal to -4 we obtain from (49) and (50)

$$W(d^{n-1}sp) = A' - n^2 + 2n - 4 + 4S(S+1). \tag{51}$$

Then letting

$$A' = n^2 - 2n + 5, \tag{52}$$

we have

$$W(d^{n-1}sp) = (2S+1)^2. \tag{53}$$

From (46) and (52)

$$S' = A' - A = -4(n-1). \tag{54}$$

From (43),

$$W(d^{n-2}s^2p) = A'' + \left[\frac{n(n-2)}{4} - S_1(S_1+1) + Q''\right]C'' + \beta'' Q'' + \delta''[S(S+1) - S_1(S_1+1)]. \tag{55}$$

Here S, S_1 are the spins of $d^{n-2}s^2p$ and d^{n-2} respectively, and A'' includes the $d-s$ interaction of $d^{n-2}s^2p$, which is a constant, [4].

We let

$$\beta'' = 4,\ C'' = -4,\ \delta'' = 4.$$

Then,

$$W(d^{n-2}s^2p) = A'' - n(n-2) + 4S(S+1). \tag{56}$$

Thus, if we let

$$A'' = (n-1)^2, \tag{57}$$

then

$$W(d^{n-2}s^2p) = (2S+1)^2. \tag{58}$$

From (46) and (57) we have

$$S'' = A'' - A = -4n. \tag{59}$$

Thus, the following parameters are used in this check:

$$\begin{aligned} A &= (n+1)^2 \\ S' &= -4(n-1) \\ S'' &= -4n \\ C &= C' = C'' = -4 \\ G'_{ds} &= -4 \\ \beta &= \beta' = \beta'' = 4 \\ \delta &= \delta' = \delta'' = 4. \end{aligned} \tag{60}$$

The expected eigenvalues then equal $(2S+1)^2$, where S is the net spin of the term considered in either $d^n p$, $d^{n-1}sp$ or $d^{n-2}s^2p$.

For $(d+s)^6 p$ we must be particularly careful to modify the heights. Since the matrices of C, β, and δ are the same for d^6p as for d^4s^2p we have here

$$W(d^6p) = W(d^4s^2p). \tag{61}$$

However, from (57)

$$A''(d^4s^2p) = 25.$$

Thus,

$$A(d^6p) = 25.$$

Also, from (52)

$$A'(d^5sp) = 29.$$

Thus, the heights of $(d+s)^6 p$ are given by

$$\begin{aligned} A(d^6p) &= 25 \\ S' &= 4 \\ S'' &= 0. \end{aligned} \tag{62}$$

In the configuration $(d+s)^2 p$, the coefficient of δ'' is a constant. Since the matrix of δ'' is not put on tape, it must be incorporated into S''.

For s^2p,

$$S(S+1) - S_1(S_1+1) = 3/4.$$

Then,

$$S''(s^2p) = -8 + (3/4)(4) = -5.$$

5. The Check T and T_c

The configurations d^n for $n < 5$ are characterized by the parameter T [1], whereas the configurations d^n for $n > 5$ are characterized by T_c [2]. The configuration d^5 can be considered as either con-

sisting of 5 electrons d or as 5 holes in the d shell. In the former case the parameter needed is T, whereas in the latter case the parameter T_c is to be used. The relationship between these two cases is given by (65), [10].

An interesting and very useful relationship was obtained by Racah [11] for the configuration d^n. If we let

$$\begin{aligned} C &= -80 \\ \alpha &= 15 \\ \beta &= 160 \\ T &= 1 \\ T_c &= 1 \end{aligned} \tag{63}$$

then,

$$W(d^n) = 240 - 40(n-2)(n-3). \tag{64}$$

Since the matrices of T and T_c for the configurations $d^n p$, $d^{n-1}sp$ and $d^{n-2}s^2p$ are just those of d^n, d^{n-1} and d^{n-2}, the equations (63) and (64) can be used to check the matrices of these parameters in the configurations $(d+s)^n p$. It should be noted that although T and T_c are parameters representing three-body interactions, [1], the combination $T+T_c$ represents two-body interactions, the same as the interactions represented by C, α, and β, [1].

Incidentally, the matrices of C, α, and β are checked here. However, as they enter other checks as well, the primary purpose here is to verify that the matrices of T and T_c are correct.

6. The Checks of the Spin-Orbit Interaction Matrices

In order to check the matrices of either ζ_d or ζ_p it is sufficient to diagonalize them separately giving a value of zero to all the other parameters. The expected eigenvalues may be obtained by determining the possible states of the configuration considered in $j-j$ coupling.

Since for an electron d, j can take on the values 3/2 or 5/2

$$\begin{aligned} (\mathbf{l}\cdot\mathbf{s}) &= \tfrac{1}{2}\left[j(j+1) - l(l+1) - s(s+1)\right] \\ &= 1, \text{ for } d_{5/2} \\ &= -\tfrac{3}{2}, \text{ for } d_{3/2}. \end{aligned} \tag{65}$$

The eigenvalues for d^n are then obtained by determining the combinations of $d_{3/2}$ and $d_{5/2}$ making up d^n.

For an electron p, j has the values 1/2 and 3/2.

Thus,

$$\begin{aligned} (\mathbf{l}\cdot\mathbf{s}) &= \tfrac{1}{2}, \text{ for } p_{3/2} \\ &= -1, \text{ for } p_{1/2}. \end{aligned} \tag{66}$$

In order to determine the multiplicity of each eigenvalue for a particular J, it is necessary to know what multiplicities of J appear in the configuration $[d_{5/2}]^k [d_{3/2}]^{n-k}$. The resulting J of $[d_{5/2}]^k$ and $[d_{3/2}]^{n-k}$, with their respective multiplicities are given on p. 263, TAS [5]. By using this information it is easy to obtain the multiplicities of the eigenvalues of the matrix ζ_d in the configuration d^n, and thus the multiplicities of the eigenvalues of ζ_d if an s and/or a p electron is added to d^n.

If ζ_p is given a value of 2 and all the other parameters equal zero, then the eigenvalues of $d^n p$, $d^{n-1}sp$ and $d^{n-2}s^2p$ are just 1 and -2 by (66). For a particular J, the multiplicity of the eigenvalue 1 equals the number of times the electron p is coupled with j equal 3/2 to the core d^n, $d^{n-1}s$, and $d^{n-2}s^2$, respectively. Similarly, the multiplicity of the eigenvalue -2 equals the number of times the electron p is coupled with j equal 1/2 to the core d^n, $d^{n-1}s$ and $d^{n-2}s^2$, respectively, for the particular J considered.

In our case we also combined the checks of ζ_d and ζ_p by letting ζ_d equal 20 and ζ_p equal 2.

The expected eigenvalues and their respective multiplicities are obtained by the same considerations as above.

Tables 7, 10, 13, 16, and 19 give the eigenvalues and their respective multiplicities for the configurations $(d+s)^n p, 2 \leq n \leq 6$ when ζ_d equals 20.

Tables 8, 11, 14, 17, and 20 give the eigenvalues and their respective multiplicities for the configurations $(d+s)^n p, 2 \leq n \leq 6$ when ζ_p equals 2.

Tables 6, 9, 12, 15, 18, and 21 give the eigenvalues and their respective multiplicities for the configurations $(d+s)^n p, 1 \leq n \leq 6$, in the combined check with ζ_d equal to 20 and ζ_p equal to 2.

It should be noted that in tables 7, 10, 13, 16, and 19 the column "Eigenvalue" gives the eigenvalues of s^2p,sp and p when ζ_p has a value of 2. These values are necessary in order to obtain the multiplicities of the eigenvalues for the case of ζ_p equal to 2, and for the combined check ζ_d equal to 20 and ζ_p equal to 2.

For five electrons d the combination $(3/2)^5$ does not occur because for j equal to 3/2 there are only 4 values of m_j and by Pauli's principle it is forbidden for two electrons to have the same values of $(nljm_j)$.

For $(d+s)^6 p$ only the pertinent values for the configuration d^5sp are required since d^6p and d^4s^2p are equivalent to d^4p here.

TABLE 6. *Eigenvalues and their multiplicities for the combined* ζ_d, ζ_p *Check of* (d+s)p

Conf.	J / Eigenvalue	0	1	2	3	4
sp	1		1	1		
	−2	1	1			
dp	21		1	1	1	1
	18			1	1	
	−29	1	1	1	1	
	−32		1	1		

TABLE 7. *Eigenvalues and their multiplicities for the ζ_d Check of* $(d+s)^2p$

n	Eigenvalue ζ_{dn}	J Combinations	J_{dn}	$J_{s2}-n_p$	Eigenvalue	J total
						$\frac{1}{2}$ $\frac{3}{2}$ $\frac{5}{2}$ $\frac{7}{2}$ $\frac{9}{2}$ $\frac{11}{2}$
0	0			3/2	1	1
	0			1/2	−2	1
1	20	5/2	5/2	2	1	
				1	1	1 3 4 3 1
				1	−2	
	−30	3/2	3/2	0	−2	3 4 3 1
2	40	$(5/2)^2$	0 2 4			2 3 3 3 2 1
				$\frac{3}{2}$	1	
	−10	(5/2)(3/2)	1 2 3 4			3 5 6 5 3 1
				$\frac{1}{2}$	−2	
	−60	$(3/2)^2$	0 2			2 3 2 1

TABLE 8. *Eigenvalues and their multiplicities for the ζ_p Check of* $(d+s)^2p$

Conf.	J / Eigenvalue	1/2	3/2	5/2	7/2	9/2	11/2
s^2p	1		1				
	−2	1					
dsp	1	3	4	4	3	1	
	−2	1	3	3	1		
d^2p	1	4	7	7	6	3	2
	−2	3	4	4	3	2	

TABLE 9. *Eigenvalues and their multiplicities for the combined ζ_d, ζ_p Check of* $(d+s)^2p$

Conf.	J / Eigenvalue	1/2	3/2	5/2	7/2	9/2	11/2
s^2p	1		1				
	−2	1					
dsp	21	1	2	2	2	1	
	18		1	2	1		
	−29	2	2	2	1		
	−32	1	2	1			
d^2p	41	1	2	2	2	1	1
	38	1	1	1	1	1	
	−9	2	3	4	3	2	1
	−12	1	2	2	2	1	
	−59	1	2	1	1		
	−62	1	1	1			

TABLE 10. *Eigenvalues and their multiplicities for the ζ_d Check of* $(d+s)^3p$

n	Eigenvalue $\zeta_d n$	J Combinations	$J_d n$	$J_s 3-n_p$	Eigenvalue	J Total
						0 1 2 3 4 5 6 7
1	20	5/2	5/2	3/2	1	1 2 2 1
	−30	3/2	3/2	1/2	−2	1 2 2 1
	40	$(5/2)^2$	0 2 4	2	1	2 5 6 6 5 3 1
2	−10	(5/2)(3/2)	1 2 3 4	1	1	3 8 11 11 8 4 1
	−60	$(3/2)^2$	0 2	1	−2	2 5 5 3 1
	60	$(5/2)^3$	3/2 5/2 9/2	0	−2	1 3 4 4 3 2 1
	10	$(5/2)^2 3/2$	1/2 $(3/2)_2$ $(5/2)_2$ $(7/2)_2$ 9/2 11/2	3/2	1	3 8 11 11 9 6 3 1
3	−40	$5/2(3/2)^2$	1/2 3/2 $(5/2)_2$ 7/2 9/2			2 6 8 8 6 3 1
	−90	$(3/2)^3$	3/2	1/2	−2	1 2 2 1

TABLE 11. *Eigenvalues and their multiplicities for the ζ_p Check of* $(d+s)^3p$

Conf.	J / Eigenvalue	0	1	2	3	4	5	6	7
ds^2p	1	1	2	2	2	1			
	−2		1	2	1				
d^2sp	1	4	11	14	13	9	5	2	
	−2	3	7	8	7	5	2		
d^3p	1	5	12	15	16	12	8	4	1
	−2	2	7	10	8	6	3	1	

TABLE 12. *Eigenvalues and their multiplicities for the combined ζ_d, ζ_p Check of* $(d+s)^3p$

Conf.	J / Eigenvalue	0	1	2	3	4	5	6	7
ds^2p	21		1	1	1	1			
	18			1	1				
	−29	1	1	1	1				
	−32		1	1					
d^2sp	41	1	3	4	4	3	2	1	
	38	1	2	2	2	2	1		
	−9	2	5	7	7	5	3	1	
	−12	1	3	4	4	3	1		
	−59	1	3	3	2	1			
	−62	1	2	2	1				
d^3p	61	1	2	2	3	2	1	1	
	58		1	2	1	1	1		
	11	2	5	7	7	6	4	2	1
	8	1	3	4	4	3	2	1	
	−39	1	4	5	5	4	2	1	
	−42	1	2	3	3	2	1		
	−89	1	1	1	1				
	−92		1	1					

TABLE 13. *Eigenvalues and their multiplicities for the ζ_d Check of* $(d+s)^4p$

n	Eigenvalue d^n	J Combinations	$J_d n$							$J_d 4-n_p$	Eigenvalue	J Total							
												1/2	3/2	5/2	7/2	9/2	11/2	13/2	15/2
2	40	$(5/2)^2$	0		2		4			3/2	1	2	3	3	3	2	1		
	−10	(5/2) 3/2		1	2	3	4			1/2	−2	3	5	6	5	3	1		
	−60	$(3/2)^2$	0		2							2	3	2	1				
3	60	$(5/2)^3$		3/2	5/2		9/2			2	1	4	7	8	7	5	3	1	
	10	$(5/2)^2$ 3/2	1/2	$(3/2)_2$	$(5/2)_2$	$(7/2)_2$	(9/2)	11/2		1	1	11	19	22	20	15	9	4	1
	−40	5/2 $(3/2)^2$	1/2	3/2	$(5/2)_2$	7/2	9/2			1	−2	8	14	16	14	9	4	1	
	−90	$(3/2)^3$		3/2						0	−2	3	4	3	1				
4	80	$(5/2)^4$	0		2		4					2	3	3	3	2	1		
	30	$(5/2)^3$ 3/2	0	1_2	2_2	3_2	4_2	5	6	3/2	1	7	12	14	13	10	6	3	1
	−20	$(5/2)^2(3/2)^2$	0_2	1	2_2	3_2	4_2	5	6	1/2	−2	8	14	16	15	11	7	3	1
	−70	5/2 $(3/2)^3$		1	2	3	4					3	5	6	5	3	1		
	−120	$(3/2)^4$	0									1	1						

TABLE 14. *Eigenvalues and their multiplicities for the ζ_p Check of* $(d+s)^4p$

Conf.	J / Eigenvalue	1/2	3/2	5/2	7/2	9/2	11/2	13/2	15/2
d^2s^2p	1	4	7	7	6	3	2		
	−2	3	4	4	3	2			
d^3sp	1	17	27	31	28	19	11	5	1
	−2	9	17	18	14	10	5	1	
d^4p	1	12	23	25	23	17	11	4	2
	−2	9	12	14	13	9	4	2	

TABLE 15. *Eigenvalues and their multiplicities for the combined ζ_d, ζ_p Check of* $(d+s)^4p$

Conf.	*J* / Eigenvalue	1/2	3/2	5/2	7/2	9/2	11/2	13/2	15/2
d^2s^2p	41	1	2	2	2	1	1		
	38	1	1	1	1	1			
	−9	2	3	4	3	2	1		
	−12	1	2	2	2	1			
	−59	1	2	1	1				
	−62	1	1	1					
d^3sp	61	3	4	5	5	3	2	1	
	58	1	3	3	2	2	1		
	11	7	12	14	13	10	6	3	1
	8	4	7	8	7	5	3	1	
	−39	5	9	10	9	6	3	1	
	−42	3	5	6	5	3	1		
	−89	2	2	2	1				
	−92	1	2	1					
d^4p	81	1	2	2	2	1	1		
	78	1	1	1	1	1			
	31	4	8	9	8	7	4	2	1
	28	3	4	5	5	3	2	1	
	−19	5	9	10	10	7	5	2	1
	−22	3	5	6	5	4	2	1	
	−69	2	3	4	3	2	1		
	−72	1	2	2	2	1			
	−119		1						
	−122	1							

TABLE 16. *Eigenvalues and their multiplicities for the ζ_d Check of* $(d+s)^5p$

n	Eigenvalue $\zeta_d n$	*J* Combinations	$J_d n$							$J_d 5-n_p$	Eigenvalue	*J* Total								
												0	1	2	3	4	5	6	7	8
3	60	$(5/2)^3$		3/2	5/2		9/2					1	3	4	4	3	2	1		
	10	$(5/2)^2$ 3/2	1/2	$(3/2)_2$	$(5/2)_2$	$(7/2)_2$	9/2	11/2		3/2	1	3	8	11	11	9	6	3		1
	−40	5/2 $(3/2)^2$	1/2	3/2	$(5/2)_2$	7/2	9/2					2	6	8	8	6	3	1		
	−90	$(3/2)^3$		3/2						1/2	−2	1	2	2	1					
4	80	$(5/2)^4$	0		2		4			2	1	2	5	6	6	5	3	1		
	30	$(5/2)^3$ 3/2	0	1_2	2_2	3_2	4_2	5	6	1	1	7	19	26	27	23	16	9	4	1
	−20	$(5/2)^2(3/2)^2$	0_2	1	2_1	3_2	4_2	5	6	1	−2	8	22	30	31	26	18	10	4	1
	−70	5/2 $(3/2)^3$		1	2	3	4			0	−2	3	8	11	11	8	4	1		
	−120	$(3/2)^4$	0									1	2	1						
5	100	$(5/2)^5$			5/2								1	2	2	1				
	50	$(5/2)^4$ 3/2	1/2	$(3/2)_2$	$(5/2)_2$	$(7/2)_2$	9/2	11/2		3/2	1	3	8	11	11	9	6	3	1	
	0	$(5/2)^3(3/2)^2$	$(1/2)_2$	$(3/2)_2$	$(5/2)_4$	$(7/2)_2$	$(9/2)_2$	11/2	13/2			5	14	19	20	17	12	7	3	1
	−50	$(5/2)^2(3/2)^3$	1/2	$(3/2)_2$	$(5/2)_2$	$(7/2)_2$	9/2	11/2		1/2	−2	3	8	11	11	9	6	3	1	
	−100	5/2 $(3/2)^4$			5/2								1	2	2	1				

TABLE 17. *Eigenvalues and their multiplicities for the ζ_p Check of* $(d+s)^5p$

Conf.	J / Eigenvalue	0	1	2	3	4	5	6	7	8
d^3s^2p	1	5	12	15	16	12	7	4	1	
	−2	2	7	10	8	6	4	1		
d^4sp	1	12	35	48	48	40	28	15	6	2
	−2	9	21	26	27	22	13	6	2	
d^5p	1	7	21	28	29	25	16	9	4	1
	−2	4	11	17	17	12	8	4	1	

TABLE 18. *Eigenvalues and their multiplicities for the combined ζ_d, ζ_p Check of* $(d+s)^5$ p

Conf.	J / Eigenvalue	0	1	2	3	4	5	6	7	8
d^3s^2p	61	1	2	2	3	2	1	1		
	58		1	2	1	1	1			
	11	2	5	7	7	6	4	2	1	
	8	1	3	4	4	3	2	1		
	−39	1	4	5	5	4	2	1		
	−42	1	2	3	3	2	1			
	−89	1	1	1	1					
	−92		1	1						
d^4sp	81	1	3	4	4	3	2	1		
	78	1	2	2	2	2	1			
	31	4	12	17	17	15	11	6	3	1
	28	3	7	9	10	8	5	3	1	
	−19	5	14	19	20	17	12	7	3	1
	−22	3	8	11	11	9	6	3	1	
	−69	2	5	7	7	5	3	1		
	−72	1	3	4	4	3	1			
	−119		1	1						
	−122	1	1							
d^5p	101		1	1	1	1				
	98			1	1					
	51	2	5	7	7	6	4	2	1	
	48	1	3	4	4	3	2	1		
	1	3	9	12	13	11	8	5	2	1
	−2	2	5	7	7	6	4	2	1	
	−49	2	5	7	7	6	4	2	1	
	−52	1	3	4	4	3	2	1		
	−99		1	1	1	1				
	−102			1	1					

TABLE 19. *Eigenvalues and their multiplicities for the ζ_d Check of* d^5sp

Eigenvalue ζ_d	J Combinations	Jd^5	J_{sp}	Eigenvalue	J Total								
					1/2	3/2	5/2	7/2	9/2	11/2	13/2	15/2	17/2
100	$(5/2)^5$	5/2	2	1	1	3	4	3	1				
50	$(5/2)^4\,3/2$	1/2 (3/2)$_2$ (5/2)$_3$ (7/2)$_3$ 9/2 11/2	1	1	11	19	22	20	15	9	4	1	
0	$(5/2)^3(3/2)^2$	(1/2)$_2$ (3/2)$_3$ (5/2)$_4$ (7/2)$_3$ (9/2)$_2$ 11/2 13/2	1	−2	19	33	39	37	29	19	10	4	1
−50	$(5/2)^2(3/2)^3$	1/2 (3/2)$_2$ (5/2)$_3$ (7/2)$_2$ 9/2 11/2	0	−2	11	19	22	20	15	9	4	1	
−100	$5/2\,(3/2)^4$	5/2			1	3	4	3	1				

TABLE 20. *Eigenvalues and their multiplicities for the ζ_p Check of* d^5sp

Eigenvalue \ J	1/2	3/2	5/2	7/2	9/2	11/2	13/2	15/2	17/2
1	15	28	34	29	20	12	5	1	
−2	28	49	57	54	41	25	13	5	1

TABLE 21. *Eigenvalues and their multiplicities for the combined ζ_d, ζ_p Check of* d^5sp

Eigenvalue \ J	1/2	3/2	5/2	7/2	9/2	11/2	13/2	15/2	17/2
101	1	2	2	2	1				
98		1	2	1					
51	7	12	14	13	10	6	3	1	
48	4	7	8	7	5	3	1		
1	12	21	25	24	19	13	7	3	1
−2	7	12	14	13	10	6	3	1	
−49	7	12	14	13	10	6	3	1	
−52	4	7	8	7	5	3	1		
−99	1	2	2	2	1				
−102		1	2	1					

7. Checks for the Relative Phases of the Electrostatic and Spin-Orbit Interactions

Since the matrices of the electrostatic interaction and the spin-orbit interaction do not generally commute, it is difficult to find values of the parameters which yield simple eigenvalues. The matrices which do commute are the electrostatic matrices of $(d+s)^n$ and the matrices of ζ_p.

Now, if we check the relative phases of the electrostatic matrices of $(d+s)^n$ and the matrices of ζ_p, we also know that the relative phases of the electrostatic matrices of $(d+s)^n$ and ζ_d are correct. This follows from the previous check where the relative phases of the matrices of ζ_d and ζ_p are checked. The relative phases of the matrices of the interactions $d-p$ and/or $s-p$ and the electrostatic matrices of $(d+s)^n$ are checked in the Casimir [1], and the ^{1}P Checks. Thus, if in addition to the other checks, the Check $(d+s)^n+\zeta_p$ is considered, then we have a verification of the correctness of the relative phases of all the electrostatic and spin-orbit interaction matrices.

For $(d+s)^n$ the parameters of the Casimir Check, as obtained from (30) [1], are used:

$$A = 11n$$
$$S' = 9$$
$$S'' = 34 - 8n$$
$$B = B' = B'' = 1$$
$$G'_{ds} = 8 \tag{67}$$
$$\alpha = \frac{3}{2}$$
$$H = H' = \frac{2\sqrt{10}}{5}$$

We note that it is necessary to replace λ by α, since $L(L+1)$ refers to the configuration $(d+s)^n$ here.

To ζ_p a value of 2 is given as before.

The expected eigenvalues for each term of $(d+s)^n$ are given in [1], (pp. 33-35). Then to find the eigenvalues of $(d+s)^n p$ we simply add to the eigenvalues of $(d+s)^n$ a value of 1 or -2 depending upon whether the electron p is coupled as j equal to 3/2 or j equal to 1/2 respectively, for a particular level of $(d+s)^n p$.

As an example consider the expected eigenvalues of $(d+s)^2 p$ for J equal to 1/2. From [1] the expected eigenvalues can only have the values 57, 54, 33, 30, 21, and 18. The eigenvalue 57 can only be obtained from

$$(d+s)^2\ {}^1D_2 + p_{3/2}.$$

The eigenvalue 54 can be obtained from

$$(d+s)^2\ {}^1S_0 + p_{1/2}.$$

The eigenvalue 33 has a multiplicity of 5:

$$(d+s)^2\ {}^3P_{1,2} + p_{3/2},\ (d+s)^2\ {}^3D_{1,2} + p_{3/2},\ (d+s)^2\ {}^3F_2 + p_{3/2}.$$

The eigenvalue 30 has a multiplicity of 3:

$$(d+s)^2\ {}^3P_{0,1} + p_{1/2},\ (d+s)^2\ {}^3D_1 + p_{1/2}.$$

The eigenvalue 21 can only be obtained from

$$(d+s)^2\ {}^1D + p_{3/2}.$$

The eigenvalue 18 can be obtained from

$$(d+s)^2\ {}^1S + p_{1/2}.$$

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Publications of the National Bureau of Standards*

Selected Abstract

Danos, M., **One physicist's view of special functions** (Proc. SIAM National Meeting, Washington, D.C., June 10-12, 1969), Chapter in *Studies in Applied Mathematics* **6**, pp. 18-29 (1970).

Key words: Analytical continuation; electron scattering; potential scattering; recoupling coefficients; Regge poles; special functions.

The need for presently unavailable information is illustrated with three examples taken from physics problems of present interest. They involve both the evaluation of integrals over special functions and the analytic continuation in one or several variables of diverse quantities which can be defined by differential or difference equations. Of particular interest would be the solution of the Schrodinger equation for the scattering of a particle in a Yukawa potential.

Other NBS Publications

J. Res. Nat. Bur. Stand. (U.S.), 75A (Phys. & Chem.), No. 6 (Nov.-Dec. 1971), SD Catalog No. C13.22/sec.A:75/6.

The accuracy of measurements of viscosity of liquids. Robert S. Marvin.

An absolute determination of viscosity using a torsional pendulum. Hobart S. White and Elliot A. Kearsley.

An absolute determination of viscosity using channel flow. Robert W. Penn and Elliot A. Kearsley.

A new determination of the atomic weight of zinc. George Marinenko and Robert T. Foley.

High-speed (subsecond) measurement of heat capacity, electrical resistivity, and thermal radiation properties of niobium in the range 1500 to 2700 K. Ared Cezairliyan.

Stable radical-anions derived from glyoxal bis(phenylhydrazones. Alexander J. Fatiadi.

Temperature dependence of photocurrents produced by x and gamma rays in silicon radiation detectors. Karl Scharf and Robert K. Mohr.

Isomerization processes in ions of the empirical formula $C_4H_8^+$. S. G. Lias and P. Ausloos.

The photochemistry of propane at high photon energies (8.4-21.2 eV). R. E. Rebbert, S. G. Lias, and P. Ausloos.

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The absorption spectra of krypton and xenon in the wavelength range 330-600 Å. K. Codling and R. P. Madden.

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Thermal analysis of calcium sulfate dihydrate and supposed α and β forms of calcium sulfate hemihydrates from 25 to 500 °C. James R. Clifton.

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